Improper Integrals

So far, definite integrals have been used to compute areas of *finite regions*. It is possible to extend the notion of area to regions lying under the graph of a function over infinite intervals, and to functions which have vertical asymptotes.

Definition: Improper Integrals of Type I are defined to be those of the form:

$$\int_a^\infty f(x)dx = \lim_{T\to\infty}\int_a^T f(x)dx$$

$$\int_{-\infty}^a f(x)dx = \lim_{T\to-\infty}\int_T^a f(x)dx$$

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx, \textit{ if both limits exist.}$$

Example: $\displaystyle\int_1^\infty x^{-2}dx = \lim_{T\to\infty}\int_1^T x^{-2}dx = \lim_{T\to\infty}\frac{x^{-1}}{-1}\Big|_1^T = \lim_{T\to\infty}\frac{-1}{x}\Big|_1^T =$

$$\lim_{T\to\infty}\left(\frac{-1}{T} - \frac{-1}{1}\right) = 1 - \lim_{T\to\infty}\frac{1}{T} = 1 - 0 = 1$$

Example: $\displaystyle\int_1^\infty x^{-1}dx = \lim_{T\to\infty}\int_1^T x^{-1}dx = \lim_{T\to\infty}\ln|x||_1^T =$

$$\lim_{T\to\infty}\ln T - \ln 1 = \infty$$

Example: $\displaystyle\int_{-\infty}^{-1} x^{-1}dx = \lim_{T\to-\infty}\int_{-\infty}^{-1} x^{-1}dx =$

$$\lim_{T\to-\infty}\ln|x||_T^{-1} = \lim_{T\to-\infty}\ln|-1| - \ln|T| = -\infty$$

If the limit defining an improper integral exists, we say that the integral **converges** or is **convergent**, otherwise we say it **diverges** or is

divergent.

Example: If s is considered to be a constant,

$$\int_0^\infty e^{-sx}dx = \lim_{T\to\infty}\int_0^T e^{-sx}dx = \lim_{T\to\infty}\frac{1}{-s}e^{-sx}\Big|_0^T =$$

$$\lim_{T\to\infty}\frac{1}{-s}e^{-s(T)} - \frac{1}{-s}e^{-s(0)} = \frac{1}{s} - \lim_{T\to\infty}\frac{1}{-s}e^{-sT} = \frac{1}{s}$$

Definition: If f is defined on the interval $0, \infty)$, and treating s as a constant in integrating, we can define a function of $F(s)$ of s by letting

$$F(s) = \int_0^\infty e^{-sx}f(x)dx = \lim_{T\to\infty}\int_0^T f(x)e^{-sx}dx$$

This function is called the **Laplace Transform** of f. In the previous example we showed that the Laplace transform of the constant function 1 is $\frac{1}{s}$. Most of the integration techniques learned in this course will be used far more often in practical applications of Calculus via the Laplace transform than in the computation of areas.

Example:(closely related to Example 2, p.489 of brown Stew

$$\int_0^\infty xe^{-sx}dx = \lim_{T\to\infty}\int_0^T xe^{-sx}dx$$

We integrate $\int xe^{-sx}dx$ by parts with $u = x$, $dv = e^{-sx}dx$, so that $du = dx$ and $v = -\frac{1}{s}e^{-sx}$:

$$\int xe^{-sx}dx = \int udv = uv - \int vdu = x\left(-\frac{1}{s}e^{-sx}\right) - \int -\frac{1}{s}e^{-sx}dx =$$

$$-\frac{x}{s}e^{-sx} + \frac{1}{s}\int e^{-sx}dx = -\frac{x}{s}e^{-sx} - \frac{1}{s^2}e^{-sx}$$

so

$$\lim_{T\to\infty}\int_0^T xe^{-sx}dx = \lim_{T\to\infty}\left(-\frac{x}{s}e^{-sx} - \frac{1}{s^2}e^{-sx}\right)\Big|_0^T =$$

$$\lim_{T\to\infty}\left(-\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-sT}\right) - \left(-\frac{0}{s}e^{-sx} - \frac{1}{s^2}e^{-s(0)}\right) = \frac{1}{s^2}$$

Note: Examples 3 & 4 of *brown Stewart* 7.9 are very important:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{T\to-\infty} \int_{T}^{0} \frac{dx}{1+x^2} + \lim_{T\to\infty} \int_{0}^{T} \frac{dx}{1+x^2} =$$

$$\lim_{T\to-\infty} \arctan x\,|_{T}^{0} + \lim_{T\to\infty} \arctan x\,|_{0}^{T} =$$

$$\lim_{T\to-\infty} (\arctan 0 - \arctan T) + \lim_{T\to\infty} (\arctan T - \arctan 0) =$$

$$0 - \lim_{T\to-\infty} \arctan T + \lim_{T\to\infty} \arctan T - 0 =$$

$$\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{T\to\infty} \int_1^T x^{-p} dx = \lim_{T\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^T =$$

$$\lim_{T\to\infty} \frac{T^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} =$$

$$\frac{1}{p-1} + \lim_{T\to\infty} \frac{T^{1-p}}{1-p} =$$

$$\frac{1}{p-1} \text{ if and only if } p > 1$$

The other type of improper integral occurs when the integrand $f(x)$ has a vertical asymptote at say $x = c$.

Then, assuming $a < c < b$, we define

$$\int_a^c f(x)dx = \lim_{T \to c^-} \int_a^T f(x)dx \qquad \int_c^b f(x)dx = \lim_{T \to c^+} \int_T^b f(x)dx$$

and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ (if both integrals exist).}$$

Comparison Theorems: If we have $f(x) \leq g(x)$ on the (possibly infinite) interval (a,b),

then we know that $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$. Thus we can conclude for proper integrals that

if $\int_a^b f(x)dx$ diverges, then so must $\int_a^b g(x)dx$, and conversely, if $\int_a^b g(x)dx$ converges, then so must $\int_a^b f(x)dx$.

Example: The region $\mathcal{R} = \left\{(x, y) | x \geq 1, 0 \leq y \leq \frac{1}{x}\right\}$ is revolved about the x-axis to form a solid object, known as **Gabriel's Horn.** It volume is

$$V = \int_1^\infty \frac{\pi}{x^2} dx = \pi \lim_{T \to \infty} \int_1^T x^{-2} dx = \pi \lim_{T \to \infty} \frac{x^{-1}}{-1} \bigg|_1^T = \pi \lim_{T \to \infty} \frac{-1}{x} \bigg|_1^T = \pi \lim_{T \to \infty} \frac{-1}{T} - \frac{-1}{1} = \pi$$

However, as we shall soon see, the formula for its surface area is

$$S = \int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx \geq \int_1^\infty 2\pi \frac{1}{x} = \infty \text{ !!!}$$

Thus we now have a pattern for a paint bucket whose volume is finite, but which cannot be painted because its surface area is infinite!