# Inverse Functions

Recall the **Vertical Line Test** which tells us whether or not a curve can be the graph of a function:

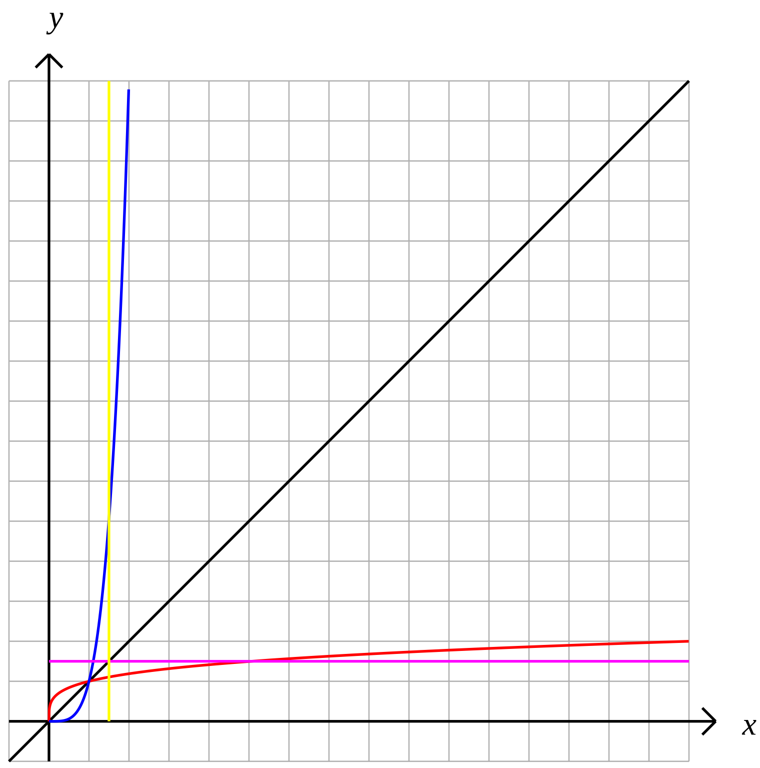
If no vertical line intersects the curve more than once, then the curve is the graph of a function.

There is also the **Horizontal Line Test** :

If no horizontal line intersects the curve more than once, then the curve is the graph of a one-to-one function.

One-to-one functions are functions which do not achieve any value more than once on a specified interval. Any function which is strictly increasing or strictly decreasing on an interval is one-to-one on that interval.

Since the reflection of the graph of a one-to-one function $f$ in the line $y = x$ clearly passes the Vertical Line Test, it is the graph of a new function, which we call the **inverse** of $f$ and which we denote by $f^{-1}$ or by $f^{inv}$. The domain of $f^{-1}$ is the range of $f$ and vice-versa.




It is best to look at a Java Applet .

We always havethe so-called **Cancellation Equations:**

$f^{-1}(f(x)) = x$ for all $x$ in the domain of $f$,

and $f(f^{-1}(y)) = y$ for all $y$ in the range of $f$.

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**Theorem:** The inverse function of a one-to-one continuous function is also continuous.

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Differentiation of the Cancellation Equations (using the Chain Rule) yields useful information about the derivatives of inverse functions, when they exist:

$$\frac{d}{dy}\left(f(f^{-1}(y))\right) = \frac{d}{dy}(y) = 1 \iff$$

$$f'(f^{-1}(y))\frac{d}{dy}\left(f^{-1}(y)\right) = 1 \iff$$

$$\frac{d}{dy}\left(f^{-1}(y)\right) = \frac{1}{f'(f^{-1}(y))}$$

Note that there is a problem if the denominator is 0.

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or

$$\frac{d}{dy}\left(f(f^{inv}(y))\right) = \frac{d}{dy}(y) = 1 \iff$$

$$f'(f^{inv}(y))\frac{d}{dy}\left(f^{inv}(y)\right) = 1 \iff$$

$$\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{f'(f^{inv}(y))}$$

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**Examples:** (1) $f(x) = 2x + 1$, so $f'(x) = 2$, therefore $\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{f'(f^{inv}(y))} = \frac{1}{2}$

(2) $f(x) = x^3$, so $f'(x) = 3x^2$, and thus

$$\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{f'(f^{inv}(y))} = \frac{1}{3\,(f^{inv}(y))^2}$$

which is rather unsatisfactory as an answer. We have to find a formula for $f^{inv}(y)$ to complete the calculation:

Using the cancellation equation $f(f^{inv}(y)) = y$, we have $\left(f^{inv}(y)\right)^3 = y$, so $f^{inv}(y) = y^{\frac{1}{3}}$.

Therefore

$$\frac{d}{dy}\left(f^{inv}(y)\right) = \frac{1}{3\,(f^{inv}(y))^2} = \frac{1}{3\left(y^{\frac{1}{3}}\right)^2} = \frac{1}{3}y^{-\frac{2}{3}}$$

In most cases, it is impossible to explicitly calculate a formula for the inverse function. We will look at a number of extremely important cases where this can be done.

# Natural Logarithms

Since $f(x) = \frac{1}{x}$ is continuous on the interval $(0, \infty)$, the function $F(x)$ which we define to be equal to the definite integral

$F(x) = \int_1^x \frac{1}{t} dt$ is differentiable for all $x > 0$.

Its derivative is $F'(x) = \frac{1}{x} > 0$, so the graph of $F$ is strictly monotone increasing, and therefore one-to-one.
Its second derivative is $F'(x) = -\frac{1}{x^2} < 0$, so the graph is concave down.

We also have $F(1) = 0$.

For reasons that will soon become clear, $F(x)$ is usually denoted by $\ln x$ and is called the **natural logarithm** function.

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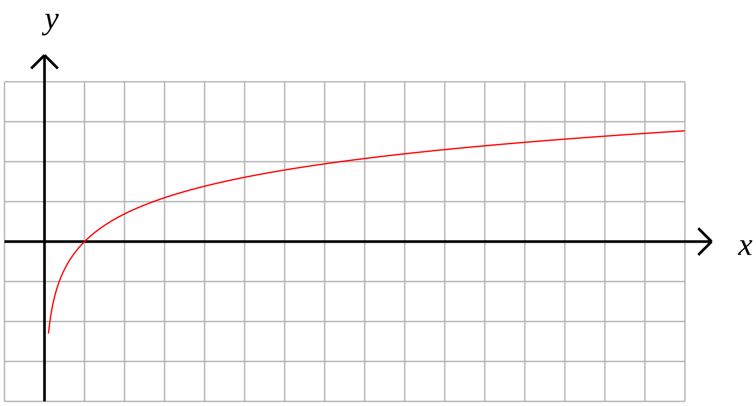
We know that the graph looks like:

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Let $h(x) = F(ax)$, where $a > 0$ is a constant.

Then $h'(x) = F'(ax)\frac{d}{dx}(ax) = \frac{1}{ax}a = \frac{1}{x}$,

so $F(ax)$ and $F(x)$ have the same derivative, and therefore $F(ax) - F(x)$ has derivative 0 and is thus a constant function.

Letting $x = 1$, we get $F(a(1)) - F(1) = F(a) - 0 = F(a)$,

so we must have $F(ax) - F(x) = F(a)$, or $F(ax) = F(a) + F(x)$ for all $x > 0$.
Using the $\ln x$ notaation for $F(x)$, we get

$\ln(ax) = \ln a + \ln x$, or, replacing $x$ by $b$, we get the familiar identity for logarithms:

$$\ln(ab) = \ln a + \ln b$$

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Similarly we can show that

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b \text{ and } \ln\left(a^b\right) = b \ln a$$

---

We can use the last property to show that

$\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to\infty} \ln x = \infty$ :

We know that $\ln 2 > 0$, so $\ln 2^x = x \ln 2$. But this appoaches $\infty$ as $x$ approaches $\infty$, and approaches $-\infty$ as $x$ approaches $-\infty$.

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When we differentiate the logarithm of a function $f(x)$, something very important happens: we get the **relative rate of change** of the function:

$$\frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)}$$

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# Logarithmic Differentiation

One of the most important uses of the natural logarithm function is in the computation of derivatives of functions which are made up of products, quotients and powers of more elementary functions. We use the three basic arithmetic properties of the logarithm to simplify the function.

**Example:** Find $y'$ if $y = \dfrac{x^4\pi^x \sin^5(3x)}{(x-1)^3(x+2)^{7.5}}$

Taking logarithms of both sides of the equation, we get

$\ln y = \ln\left(\dfrac{x^4\pi^x \sin^5(3x)}{(x-1)^3(x+2)^{7.5}}\right)$ or

$\ln y = 4\ln x + x\ln\pi + 5\ln\sin(3x) - 3\ln(x-1) - 7.5\ln(x+2)$

which we now differentiate:

$$\frac{y'}{y} = 4\frac{1}{x} + \ln\pi + 5\frac{3\cos(3x)}{\sin(3x)} - 3\frac{1}{x-1} - 7.5\frac{1}{x+2}$$

which we need only simplify slightly to get $y'$ in a usable form:

$$y' = y\left[\frac{4}{x} + \ln\pi + 15\cot(3x) - \frac{3}{x-1} - \frac{7.5}{x+2}\right]$$

# Negative $x$

There will be occasions when we wish to apply logarithms and deal with negative values of the variables concerned.

Of course, $\ln x$ is undefined if $x \le 0$. However, $\ln|x|$ *is* defined if $x < 0$.

Let us then find the derivative of $\ln|x|$ for non-zero $x$.

If $x > 0$, it is of course $\dfrac{1}{x}$.

If $x < 0$, then $|x| = -x$, so $\ln|x| = \ln(-x)$, and we can apply the Chain Rule:

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{1}{-x}(-1) = \frac{1}{x},$$

so we have the important formula

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ if } x \neq 0$$

# The Exponential Function

Since $F(x) = \ln x$ is one-to-one on $(0, \infty)$ and has range $(-\infty, \infty)$, it has an inverse function $F^{inv}$ which is called the **exponential function** which has domain $(-\infty, \infty)$ and range $(0, \infty)$.

It is usually written as $e^x$, and the Cancellation Laws give us

$e^{\ln x} = x$ and $\ln(e^x) = x$ .

The exponential function inherits its important arithmetic properties from $\ln x$:

$e^{a+b} = e^a e^b$ , $e^{a-b} = \dfrac{e^a}{e^b}$ , and $(e^a)^b = e^{ab}$ .

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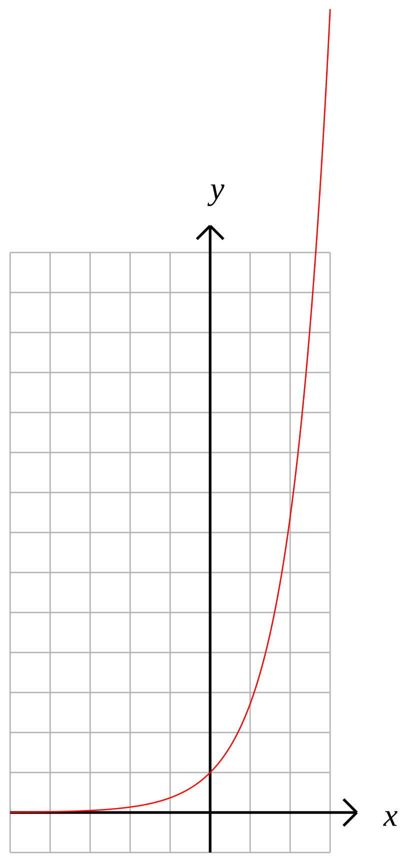
Differentiation of the equation $\ln(e^x) = x$ gives us $\dfrac{1}{e^x}\dfrac{d}{dx}(e^x) = 1$, so we have

$\dfrac{d}{dx}(e^x) = e^x > 0$ , so $e^x$ is one of the very special functions which equals its own derivative, which is always positive.

The graph of $y = e^x$ is:

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Clearly $e^0 = 1$. The value of $e^1$ is called $e$, and equals, to 20 decimal places,

2.71828182845904523536.

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We have the important differentiation formula:

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$$

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**Example:** Sketch the graph of $y = f(x) = xe^{-x^2}$

**Solution:** We have

$$y' = x\left(e^{-x^2}\right)' + (x)'e^{-x^2} = x\left(e^{-x^2}(-2x)\right) + (1)e^{-x^2} = (1-2x^2)e^{-x^2}$$

and

$$y'' = (1-2x^2)\left(e^{-x^2}\right)' + (1-2x^2)'e^{-x^2} =$$

$$(1-2x^2)\left(e^{-x^2}(-2x)\right) + (-4x)e^{-x^2} = -2x\left(3-2x^2\right)e^{-x^2}$$

The critical numbers of $f$ are $-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$, and the inflection numbers are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$.
Since $f$ is an odd function, we only look at a table of signs for non-negative $x$:

| $x$ | $0$ | $\left(0, \frac{\sqrt{2}}{2}\right)$ | $\frac{\sqrt{2}}{2}$ | $\left(\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}\right)$ | $\sqrt{\frac{3}{2}}$ | $\left(\sqrt{\frac{3}{2}}, \infty\right)$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $f''(x)$ | 0 | $-$ | $-$ | $-$ | 0 | $+$ | $+\infty$ |
| $f'(x)$ | 1 | $+$ | 0 | $-$ | $-$ | $-$ | 0 |
| $f(x)$ | 0 | $+$ | $+$ | $+$ | $+$ | $+$ | 0 |

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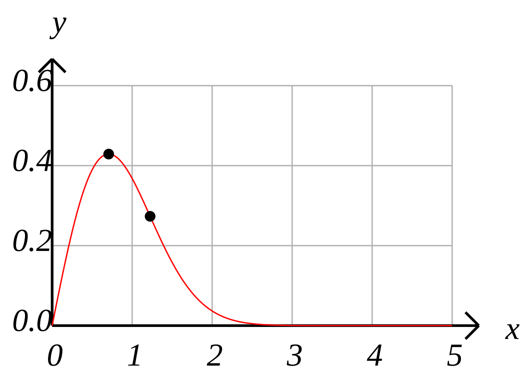
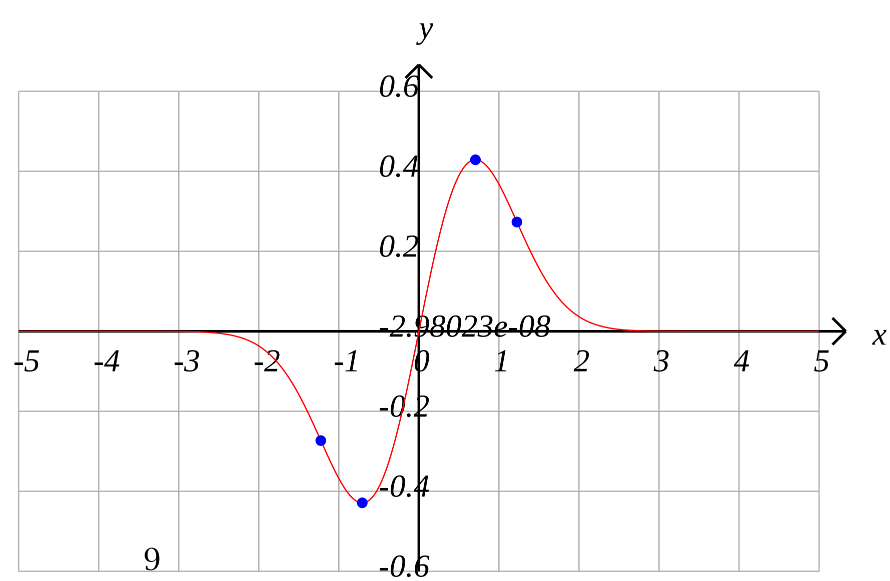
We use this to sketch a graph of $f$ for non-negative $x$:




and then we use this to sketch the whole graph:



We also have the integration formula

$$\int e^x = e^x + C$$

which may be used to evaluate many indefinite integrals using the Method of Substitution:

$$\int e^{\sin x} \cos x dx = e^{\sin x} + C$$

or

$$\int 2xe^{x^2} dx = e^{x^2} + C$$

# Generalized Logarithms and Exponentials

If $a > 0$, we define, for any real number $a > 0$, the Generalized Exponential Functions

$$f(x) = a^x := e^{(\ln a)x} > 0.$$

These functions have two important classes of properties, which are easily derived from those of $e^x$:

**Arithmetic Properties:**

$a^x a^y = a^{x+y}$, since $a^x a^y = e^{(\ln a)x} e^{(\ln a)y} = e^{(\ln a)(x+y)} = a^{x+y}$

$\dfrac{a^x}{a^y} = a^{x-y}$, since $\dfrac{a^x}{a^y} = \dfrac{e^{(\ln a)x}}{e^{(\ln a)y}} = e^{(\ln a)(x-y)} = a^{x-y}$

$(a^x)^y = a^{xy}$, since $(a^x)^y = \left(e^{(\ln a)x}\right)^y = \left(e^{((\ln a)x)y}\right) = a^{xy}$

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**Calculus Properties:**

**First Derivative:**

$(a^x)' = \left(e^{(\ln a)x}\right)' = e^{(\ln a)x}\,((\ln a)\,x)' = e^{\ln ax}\,(\ln a) = (\ln a)\,e^{\ln ax} = (\ln a)\,a^x$

or

$$(a^x)' = (\ln a)\,a^x$$

**Antiderivative:** $\int a^x dx = \dfrac{1}{\ln a} a^x + C$

**Second Derivative:**

$$(a^x)'' = (\ln a)^2\,a^x \geq 0.$$

Thus the second derivative is positive if $a \neq 1$, and the first derivative is positive is $a > 1$ and is negative if $0 < a < 1$, so we have two possible types of graph of $y = a^x$. These are best viewed with a Java applet .

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**The Inverse of $a^x$**

In either case, so long as $a > 0$ and $a \neq 1$, we have that $f(x) = a^x$ is one-to-one, with domain $(-\infty, \infty)$, and range $(0, \infty)$, so $f(x) = a^x$ has an inverse $f^{inv}$ with domain $(0, \infty)$ and range $(-\infty, \infty)$.

This inverse is usually written $\log_a x$, and is called the **logarithm to base $a$ of $x$** .

The Cancellation Equations for this inverse are:

$$a^{\log_a x} = x \text{ and } \log_a(a^x) = x$$

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Fron the defintion of $a^x = e^{(\ln a x)}$ we have $a^{\log_a x} = e^{(\ln a \log_a x)} = x$,

and since $e^x$ is one-to-one and $e^{\ln x} = x$, we must have $\ln a \log_a x = \ln x$, so we get a very useful formula for $\log_a x$ :

$$\log_a x = \frac{\ln x}{\ln a}$$

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The logarithm functions have two important classes of properties:

**Arithmetic Properties:**

$\log_a xy = \log_a x + \log_a y$ , since

$$\log_a xy = \frac{\ln xy}{\ln a} = \frac{\ln x + \ln y}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} = \log_a x + \log_a y$$

$\log_a \frac{x}{y} = \log_a x - \log_a y$ , since

$$\log_a \frac{x}{y} = \frac{\ln \frac{x}{y}}{\ln a} = \frac{\ln x - \ln y}{\ln a} = \frac{\ln x}{\ln a} - \frac{\ln y}{\ln a} = \log_a x - \log_a y$$

$\log_a x^y = y \log_a x$ , since

$$\log_a x^y = \frac{\ln x^y}{\ln a} = \frac{y \ln x}{\ln a} = y \frac{\ln x}{\ln a} = y \log_a x$$

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**Calculus Properties:**

**First Derivative:**

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x \ln a}$$

or

$$(\log_a x)' = \frac{1}{x \ln a}$$

**Second Derivative:**

$$(\log_a x)'' = -\frac{1}{x^2 \ln a}$$

Thus the signs of the first and second derivatives depend only on the sign of $\ln a$. The graphs of the logarithm functions are best viewed with a Java applet .

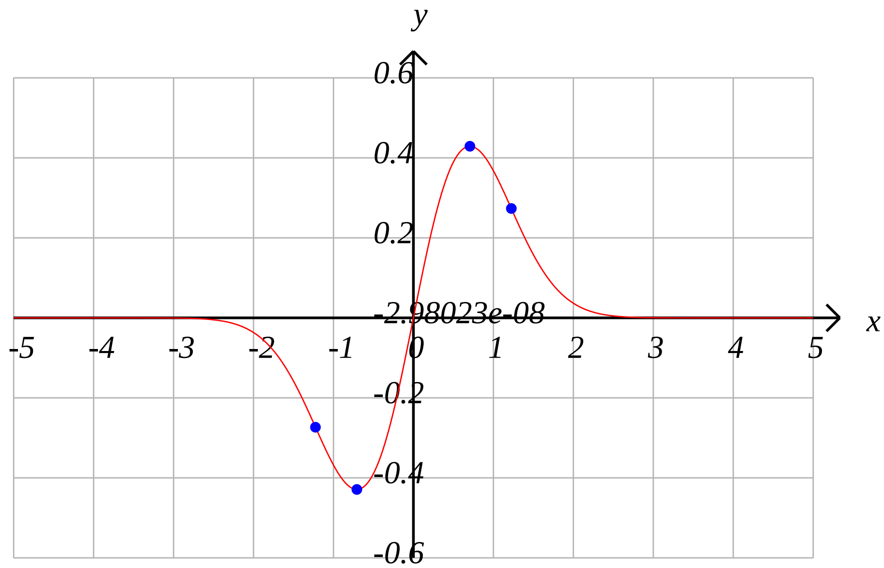
# Uses of "Non-Natural" Logarithms

The logarithm to base 10 is called the **common logarithm**. Until about 1970, it was essential for all technically skilled persons to be extremely dextrous in the use of common logarithms because without easily available electronic computing devices common logarithms were the method of choice to be used in technical computations.

Since the advent of microchip technology this dexterity is no longer needed on a daily basis by technical personnel. The natural logarithm is now preferred in the remaining extremely important use of exponential and logarithm functions: the modelling of processes characterized by **constant relative rates of change** .

There are, however, some remnants of the common logarithm still in current use: scientific measurements of magnitudes of earthquakes, sound levels, pH or acidity levels are expressed in units based on logarithms to the base 10.

Logarithms to base 2 are also frequently used in computational science in the analysis of complexity of calculations.


y
x
0.6
0.4
0.2
-2.98023e-08
-0.2
-0.4
-0.6
-5
-4
-3
-2
-1
0
1
2
3
4
5

# Exponential Growth and Decay

Recall that if $y = f(t)$, then $f'(t) = \dfrac{dy}{dt}$ is called the **rate of change** of $y$ with respect to $t$.

Another very important measure of rate of change is the **relative rate of change**

$$\frac{f'(t)}{f(t)} = \frac{y'}{y}.$$

If this quantity is a constant, say $k$, then there is a nice formula for $y$ as a function of $t$:

$y = y_0 e^{k(t-t_0)}$, where $y_0$ is the value of $y$ when $t = t_0$.

If $t_0 = 0$, the formula becomes even nicer: $y = y_0 e^{kt}$

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If $k > 0$, so that $y$ is increasing with $t$, we say that we have **exponential growth** .

If $k < 0$, so that $y$ is decreasing with $t$, we say that we have **exponential decay** .

It is usually clear from the context of a situation that the relative rate of change is a constant.

The practical problem is to translate given data into a useful equation for the quantity that is of interest.

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## Some Real World Models:

**(1) Model #1: Growth of Bacterial Cultures**

It is known that under certain conditions a population of bacteria grow at a constant relative rate.

The standard equation to represent this is

$P(t) = P(0)e^{kt}$, where $P(0)$ is the population at time 0, and $k$ is a constant that depends on the type of bacteria and its environmental conditions.

Usually, this problem arises in a context where the population value is measured at two different points in time, and it is required to use this information to predict the population values at other times.

In other words, two **Data Points** $(t_1, P_1)$ and $(t_2, P_2)$ are given, and one wishes to find the curve $P(t) = P(0)e^{kt}$ that passes through them.

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The simplest of such problems are those where the initial value of the population is given.

**Example:** A bacteria culture starts with 4000 bacteria, and in 3 hours has grown to 7000.
(1) Find a formula for the amount of bacteria after $t$ hours.
(2) How long will it take the population to double?
(3) How long will it take the population to grow by a factor of 10?

**Solution:** We have $P(0) = 4000$, and $P(3) = 7000$.

We know that the population function satisfies the equation $P(t) = P(0)e^{kt}$,

and since we have $P(0) = 4000$, we have $P(t) = 4000e^{kt}$.

We need to find $k$, so we use the other Data Point:

$P(3) = 7000 = 4000e^{k(3)} = 4000e^{3k}$.

We must solve for $k$:

We have $7000 = 4000e^{3k}$, so $\frac{7000}{4000} = e^{3k}$ or $\frac{7}{4} = e^{3k}$

Taking natural logarithms, we have:

$$\ln\frac{7}{4} = \ln e^{3k} = 3k, \text{ so } k = \frac{\ln\frac{7}{4}}{3}$$

and thus

$$P(t) = 4000e^{kt} = 4000e^{\frac{\ln\frac{7}{4}}{3}t} = 4000\left(\frac{7}{4}\right)^{\frac{t}{3}}$$

so now we have the desired formula for $P(t)$.

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There are still two unanswered questions:

(2) How long will it take the population to double?
Since $P(0) = 4000$, we need to find the value of $t$ for which $P(t) = 8000$, so we solve the equation:

$8000 = 4000e^{\frac{\ln\frac{7}{4}}{3}t}$ for $t$:

$$2 = e^{\frac{\ln\frac{7}{4}}{3}t}$$

$$\ln 2 = \frac{\ln\frac{7}{4}}{3}t$$

$3\ln 2 = \left(\ln \frac{7}{4}\right) t$

$t = \frac{3\ln 2}{\ln \frac{7}{4}} = \frac{3\ln 2}{\ln 7 - \ln 4} \doteq 3.71$

This number is called the **doubling time** of an exponential growth model, and is denoted by $\mathcal{T}_2$.

For a growth equation $y = y_0 e^{kt}$, we have $\mathcal{T}_2 = \frac{\ln 2}{k}$ .

(3) How long will it take the population to grow by a factor of 10?

Since $P(0) = 4000$, we need to find the value of $t$ for which $P(t) = 40000$, so we solve the equation:

$10000 = 4000e^{\frac{\ln \frac{7}{4}}{3}t}$ for $t$:

$10 = e^{\frac{\ln \frac{7}{4}}{3}t}$

$\ln 10 = \frac{\ln \frac{7}{4}}{3}t$

$3\ln 10 = \left(\ln \frac{7}{4}\right) t$

$t = \frac{3\ln 10}{\ln \frac{7}{4}} = \frac{3\ln 10}{\ln 7 - \ln 4}$

This number is called the **magnitude growth time** of an exponential growth model, and is denoted by $\mathcal{T}_{10}$.

For a growth equation $y = y_0 e^{kt}$, we have $\mathcal{T}_{10} = \frac{\ln 10}{k}$ .

**Model #2: Radioactive Decay**

It is known that radioactive elements decay at a rate proportional to the amount of the element present: in other words, the relative rate of change is constant, and negative.

For example, radium-226 decays in such a way that half of it disappears every 1590 years. The decay equation

$y(t) = y_0 e^{kt}$ must therefore satisfy $\frac{1}{2}y_0 = y_0 e^{k(1590)}$ so we can solve for $k$ by eliminating $y_0$ and taking logarithms:

$\frac{1}{2} = e^{1590k}$

$$\ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2 = 1590k$$

so

$$k = -\frac{\ln 2}{1590} \doteq -0.0004359$$

---

Many people prefer to have the constant $k$ positive, so they would start with the decay equation in the form

$$y(t) = y_0 e^{-kt}.$$

In either case, the resulting model equation is

$$y(t) = y_0 e^{-\frac{\ln 2}{1590}t} = y_0 2^{-\frac{t}{1590}}$$

Note that it is easier to understand the model's equation when it is written in terms of powers of 2, rather than powers of $e$. It is essential that the student be able to move easily from one form to another. The number 1590 is called the **half-life** of the decaying element, and is denoted by $\mathcal{T}_{\frac{1}{2}}$ or $\mathcal{T}_{0.5}$.

If the decay equation is $y(t) = y_0 e^{-kt}$, $(k > 0)$, then we have $\mathcal{T}_{\frac{1}{2}} = \dfrac{\ln 2}{k}$.

The length of time it takes an element to decay to one-tenth of its original value is called the **magnitude decay time** , denoted by $\mathcal{T}_{\frac{1}{10}}$ or $\mathcal{T}_{0.1}$ and is related to $k(> 0)$ by $\mathcal{T}_{\frac{1}{10}} = \dfrac{\ln 10}{k}$.

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# The General Problem

Suppose we are told that a quantity $y$ varies with time $t$ at a constant relative rate of change, and we are given two data points $(t_1, y_1)$ and $(t_2, y_2)$, from which we are to construct a model growth or decay equation which is then to be used to determine the doubling time or half-life and the magnitude growth or decay time.

Since we have two readings, we can insert then into the equation

$$y(t) = y_0 e^{kt}$$

and get two data equations in the two unknown constants $y_0$ and $k$:

$$y_1 = y_0 e^{kt_1} \text{ and } y_2 = y_0 e^{kt_2}.$$

Taking the ratios of the two equations, we get

$$\frac{y_2}{y_1} = \frac{y_0 e^{kt_2}}{y_0 e^{kt_1}} = e^{k(t_2 - t_1)}$$

and on taking logarithms we have

$\ln\left(\frac{y_2}{y_1}\right) = \ln y_2 - \ln y_1 = k(t_2 - t_1)$ so

$k = \frac{\ln y_2 - \ln y_1}{t_2 - t_1}$

We can now use this in the first data equation to find the remaining unknown constant $y_0$:

$y_1 = y_0 e^{kt_1} = y_0 e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1}$, so

$y_0 = y_1 e^{-\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1}$

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If we now put these values in the model equation,

$y(t) = y_0 e^{kt}$, we get

$y(t) = y_1 e^{-\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1} e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t}$

or

$y(t) = y_1 e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1}(t - t1)} = y_1 \left(\frac{y_2}{y_1}\right)^{\frac{t - t_1}{t_2 - t_1}}$

We could just as easily derive

$y(t) = y_2 \left(\frac{y_1}{y_2}\right)^{\frac{t - t_2}{t_1 - t_2}}$

The advantage of these two formulas is that is easy to check that they pass through the given data points.

Of course, when doing practical calculations, these equation should, and often must, be converted to base $e$ and natural logarithm format.

# Model#3 Newton's Law of Temperature Change

**Newton's Law of Temperature Change** says that the rate of change of the temperature of an object is directly proportional to the difference between its temperature, and the temperature of its surroundings, called the **ambient temperature**.

Industrial applications of this model are plentiful: any situation which requires the heating or cooling of significant quantities of materials is modeled by this law. Examples are found in steel mills, large bakeries, chemical complexes, especially breweries, etc. It is also used by metallurgical engineers in the tempering of metal products.

**The Model**

We will let the temperature of the object be $T(t)$, and since this will tend to the ambient temperature as $t \to \infty$, we will denote the ambient temperature by $T_\infty$.

We will also denote the initial temperature $T(0)$ by $T_0$.

It is important to notice that the temperature $T(t)$ does **not** have a constant proportional rate of change:

it is the function $\mathcal{D}(t) = T(t) - T_\infty$ that satisfies

$\dfrac{\mathcal{D}'(t)}{\mathcal{D}(t)} = -k$ for some constant $k > 0$.

We therefore have $\mathcal{D}(t) = D(0)e^{-kt} = (T_0 - T_\infty)e^{-kt}$, or, converting to expressions involving $T(t)$ only:

$T(t) - T_\infty = (T_0 - T_\infty)e^{-kt}$

or

$T(t) = T_\infty + (T_0 - T_\infty)\, e^{-kt} = T_\infty \left(1 - e^{-kt}\right) + T_0 e^{-kt}$

# The Real World

If we are lucky, we will know the initial temperature $T_0$ and the ambient temperature $T_\infty$.

We may, for example, want to cook a turkey, which we have brought home from the butcher thawed and at a temperature of 5°C. Leaving it in the kitchen sink, where the temperature is 20°C, while we prepare the stuffing, we observe that its temperature increases by 1°C in the 30 minutes it takes us to prepare the stuffing and to stuff the turkey. Our cookbook tells us to cook the turkey at an oven temperature of 180°C. If the turkey is considered to be cooked when it has reached the temperature of 85°C, how long will it take to cook the turkey?

**Solution:**

**Phase I:Pre-Oven**

Before the turkey is in the oven, we have $T_0 = 5$ and $T_\infty = 20$, and we also have $T(30) = 6$, so we can solve for $k$:

$$T(30) = T_\infty\left(1 - e^{-k(30)}\right) + T_0 e^{-k(30)} = 6 = 20\left(1 - e^{-k(30)}\right) + 5e^{-k(30)}$$

$$6 = 20 - 20e^{-30k} + 5e^{-30k} = 20 - 15e^{-30k}$$

$$-14 = -15e^{-30k}$$

$$e^{-30k} = \frac{-14}{-15}$$

$$-30k = \ln\left(\frac{14}{15}\right)$$

$$k = \frac{\ln 14 - \ln 15}{-30} = \frac{\ln 15 - \ln 14}{30} \doteq 0.0022997$$

**Phase II:In the Oven**

When we put the turkey in the oven, we have $T_0 = 6$, and $T_\infty = 180$, so the temperature at time $t$ will be

$$T(t) = T_\infty\left(1 - e^{-kt}\right) + T_0 e^{-kt} = 180\left(1 - e^{-\frac{\ln 15 - \ln 14}{30}t}\right) + 6e^{-\frac{\ln 15 - \ln 14}{30}t} =$$

$$180 - 174e^{-\frac{\ln 15 - \ln 14}{30}t}$$

We have to find the time $t$ when this equals 85:

$$85 = 180 - 174e^{-\frac{\ln 15 - \ln 14}{30}t} \text{ if}$$

$$-95 = -174e^{-\frac{\ln 15 - \ln 14}{30}t} \text{ if}$$

$$\frac{-95}{-174} = e^{-\frac{\ln 15 - \ln 14}{30}t} \text{ if( taking logarithms)}$$

$$\ln\left(\frac{95}{174}\right) = -\frac{\ln 15 - \ln 14}{30}t \text{ if}$$

$$\ln 95 - \ln 174 = -\frac{\ln 15 - \ln 14}{30}t \text{ if}$$

$$30\frac{\ln 95 - \ln 174}{-(\ln 15 - \ln 14)} = t \text{ if}$$

$$t = 30\frac{\ln 174 - \ln 95}{\ln 15 - \ln 14} \doteq 30\frac{0.6051784}{0.0689929} \doteq 263 \text{ minutes}$$

or about 4.4 hours.

# Three Temperature Readings

It sometimes happens that three temperature readings $(t_1, T_1)$, $(t_2, T_2)$, and $(t_3, T_3)$ are taken, and it is desired to use them to find the complete equation of the temperature of the object being studied.

We get three equations in the three unknown constants $T_0$, $T_\infty$, and $k$:

$$T_1 = T_\infty + (T_0 - T_\infty)\, e^{-kt_1}$$

$$T_2 = T_\infty + (T_0 - T_\infty)\, e^{-kt_2}$$

$$T_3 = T_\infty + (T_0 - T_\infty)\, e^{-kt_3}$$

Witout some extra information these equations are usually impossible to solve. It is usually easy to arrange for the times between the three measurements to be equal, say to $\Delta t$. Then we have $t_2 = t_1 + \Delta t$ and $t_3 = t_1 + 2\Delta t$, and it is possible to eliminate $T_0$ and $k$:

$$T_1 - T_\infty = (T_0 - T_\infty)\, e^{-kt_1}$$

$$T_2 - T_\infty = (T_0 - T_\infty)\, e^{-k(t_1+\Delta t)}$$

$$T_3 - T_\infty = (T_0 - T_\infty)\, e^{-(kt_1+2\Delta t)}$$

(A) $\dfrac{T_1 - T_\infty}{T_0 - T_\infty} = e^{-kt_1}$

(B) $\dfrac{T_2 - T_\infty}{T_0 - T_\infty} = e^{-k(t_1+\Delta t)}$

(C) $\dfrac{T_3 - T_\infty}{T_0 - T_\infty} = e^{-k(t_1+2\Delta t)}$

Now we divide equation (B) by equation (A) and equation (C) by equation (B):

$$\frac{T_2 - T_\infty}{T_1 - T_\infty} = \frac{e^{-k(t_1+\Delta t)}}{e^{-kt_1}} = e^{-k\Delta t}$$

$$\frac{T_3 - T_\infty}{T_2 - T_\infty} = \frac{e^{-k(t_1+2\Delta t)}}{e^{-k(t_1+\Delta t)}} = e^{-k\Delta t}$$

so we have

$$\frac{T_2 - T_\infty}{T_1 - T_\infty} = \frac{T_3 - T_\infty}{T_2 - T_\infty}$$

Note that $T_0$ has disappeared. Simplifying, we get:

$(T_2 - T_\infty)(T_2 - T_\infty) = (T_1 - T_\infty)(T_3 - T_\infty)$

or $T_2^2 - 2T_2T_\infty + T_\infty^2 = T_1T_3 - (T_1 + T_3)T_\infty + T_\infty^2$

or $(T_1 - 2T_2 + T_3)T_\infty = T_1T_3 - T_2^2$

or $T_\infty = \dfrac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3}$

We now use this to find:

$$e^{-k\Delta t} = \frac{T_2 - \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3}}{T_1 - \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3}} =$$

$$\frac{T_2(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2)}{T_1(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2)} =$$

$$\frac{T_1T_2 - 2T_2^2 + T_2T_3 - T_1T_3 + T_2^2}{T_1^2 - 2T_1T_2 + T_1T_3 - T_1T_3 + T_2^2} =$$

$$\frac{T_1T_2 - T_2^2 + T_2T_3 - T_1T_3}{T_1^2 - 2T_1T_2 + T_2^2} =$$

$$\frac{T_2(T_1 - T_2) + T_3(T_2 - T_1)}{(T_1 - T_2)^2} =$$

$$\frac{(T_3 - T_2)(T_2 - T_1)}{(T_2 - T_1)^2} = \frac{T_3 - T_2}{T_2 - T_1}$$

If this quantity is not positive there is no mathematical solution, which should tell us that there is a problem with the data or with the model. If this quantity is positive but is greater than 1 there is also a problem, because this would tell us that the temperature is going to tend to $\pm\infty$!

Using $e^{-k\Delta t} = \dfrac{T_3 - T_2}{T_2 - T_1}$ we can solve explicitly for $k$:

$$-k\Delta t = \ln\left(\frac{T_3 - T_2}{T_2 - T_1}\right) = \ln(T_3 - T_2) - \ln(T_2 - T_1)$$

$$k = \frac{\ln(T_2 - T_1) - \ln(T_3 - T_2)}{\Delta t}$$

Next we substitute the values of $k$ and $T_\infty$ into equation (A) to find $T_0$:

(A) $\dfrac{T_1 - T_\infty}{T_0 - T_\infty} = e^{-kt_1}$ becomes

$$\frac{T_1 - \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3}}{T_0 - \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3}} = e^{-\left(\frac{\ln(T_2 - T_1) - \ln(T_3 - T_2)}{\Delta t}\right)t_1} \text{ or}$$

$$\frac{T_1(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2)}{T_0(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2)} = e^{\ln\left(\frac{T_3 - T_2}{T_2 - T_1}\right)\frac{t_1}{\Delta t}} \text{ or}$$

$$\frac{T_1^2 - 2T_1T_2 + T_1T_3 - T_1T_3 + T_2^2)}{T_0(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2)} = \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{t_1}{\Delta t}} \text{ or}$$

$$\frac{(T_1 - T_2)^2}{T_0(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2)} = \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{t_1}{\Delta t}} \text{ or}$$

$$(T_1 - T_2)^2 \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{-t_1}{\Delta t}} = T_0(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2) \text{ or}$$

$$(T_1 - T_2)^2 \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{-t_1}{\Delta t}} + (T_1T_3 - T_2^2) = T_0(T_1 - 2T_2 + T_3) \text{ or}$$

$$T_0 = \frac{(T_1 - T_2)^2 \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{-t_1}{\Delta t}} + (T_1T_3 - T_2^2)}{T_1 - 2T_2 + T_3}$$

---

We now insert these values into the formula for $T(t)$:

$T(t) = T_\infty + (T_0 - T_\infty)\, e^{-kt}$

becomes:

$$T(t) = \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3} + \left(\frac{(T_1 - T_2)^2}{T_1 - 2T_2 + T_3}\right)\left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{t - t_1}{\Delta t}}$$

---

It is often convenient to change the notation: we let

$\Delta T_1 = T_2 - T_1$ and $\Delta T_2 = T_3 - T_2$, we get $T_2 = T_1 + \Delta T_1$, and $T_3 = T_1 + \Delta T_1 + \Delta T_2$, so

$$T_\infty = \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3} = \frac{T_1(T_1 + \Delta T_1 + \Delta T_2) - (T_1 + \Delta T_1)^2}{\Delta T_2 - \Delta T_1} =$$

$$\frac{T_1^2 + T_1\Delta T_1 + T_1\Delta T_2 - (T_1^2 + 2T_1\Delta T_1 + \Delta T_1^2)}{\Delta T_2 - \Delta T_1} = \frac{T_1(\Delta T_2 - \Delta T_1) - \Delta T_1^2}{\Delta T_2 - \Delta T_1} = T_1 - \frac{\Delta T_1^2}{\Delta T_2 - \Delta T_1}$$

$$e^{-k\Delta t} = \frac{T_3 - T_2}{T_2 - T_1} = \frac{\Delta T_2}{\Delta T_1}$$

$$T_0 = T_1 + \frac{(\Delta T_1)^2 \left( \left(\frac{\Delta T_2}{\Delta T_1}\right)^{\frac{-t_1}{\Delta t}} - 1 \right)}{\Delta T_2 - \Delta T_1}$$

$$T(t) = T_1 + \frac{(\Delta T_1)^2}{\Delta T_2 - \Delta T_1} \left[ \left(\frac{\Delta T_2}{\Delta T_1}\right)^{\frac{t-t_1}{\Delta t}} - 1 \right]$$

# Model # 4 Dilution of Chemicals

Suppose a tank contains a volume $V$ litres of fluid with a concentration $c_0$ of chemical X, and that there is an inflow into the tank of the fluid at the rate $r$ litres per minute with a conentration level of $c_{in}$ of chemical X. Assumung that the tank is continuously and thoroughly mixed, so that the concentration is uniform throughout the tank, and that fluid is drawn off at the same rate, so that the volume remains constant.

Clearly the concentration will tend to $c_{in}$, and the amount of chemical X in the tank will therefore tend to $c_{in}V$. Let $x(t)$ be the amount of chemical X, and let $c_{out}(t)$ be the concentration of chemical X in the tank at time $t$. We have $c_{out}(t) = \frac{x(t)}{V}\frac{\text{kg}}{\ell}$, and

$$x'(t) = \left[ c_{in}\frac{\text{kg}}{\ell} - c_{out}(t) \right] r\frac{\ell}{\text{min}} = \left[ c_{in}\frac{\text{kg}}{\ell} - \frac{x(t)}{V}\frac{\text{kg}}{\ell} \right] r\frac{\ell}{\text{min}}$$

$$= \left[ \frac{c_{in} - x(t)}{V} \right] r\frac{\text{kg}}{\text{min}}$$

Next we let $\mathcal{D}(t) = x(t) - c_{in}V$, so that

$$\mathcal{D}'(t) = x'(t) = \left[ \frac{c_{in} - x(t)}{V} \right] r\frac{\text{kg}}{\text{min}} = -\mathcal{D}(t)\frac{r}{V}\frac{\text{kg}}{\text{min}}$$

or, discarding units for the moment,

$$\frac{\mathcal{D}'(t)}{\mathcal{D}(t)} = -\frac{r}{V},$$

which has solution $\mathcal{D}(t) = \mathcal{D}(0)e^{-\frac{r}{V}t}$.

This can now be converted into expressions for $x(t)$ and $c(t)$:

We have $\mathcal{D}(0) = x(0) - c_{in}V = c_0V - c_{in}V = (c_0 - c_{in})V$, so

$\mathcal{D}(t) = (c_0 - c_{in})Ve^{-\frac{r}{V}t}$, and thus

$x(t) - c_{in}V = (c_0 - c_{in})Ve^{-\frac{r}{V}t}$, giving us

$x(t) = c_{in}V + (c_0 - c_{in})Ve^{-\frac{r}{V}t} = \left[c_{in} + (c_0 - c_{in})e^{-\frac{r}{V}t}\right]V$ and

$c(t) = c_{in} + (c_0 - c_{in})e^{-\frac{r}{V}t}$.

# Inverse Trigonometric Functions

The trigonometric functions are not one-to-one. By restricting their domains, we can construct one-to-one functions from them. For example, if we restrict the domain of $\sin x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have a one-to-one function which has an inverse denoted by $\arcsin x$ or $\sin^{-1} x$ .

Similarly if we restrict the domain of $\cos x$ to the interval $[0, \pi]$ we have a one-to-one function which has an inverse denoted by $\arccos x$ or $\cos^{-1} x$

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If we restrict the domain of $\tan x$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ we have a one-to-one function which has an inverse denoted by $\arctan x$ or $\tan^{-1} x$ .

---

The inverses of the other three trigonometric functions are not often used, but are defined similarly.

We always have the Cancellation Laws, which *only* hold on the appropriate domains:

$\sin(\sin^{-1} x) = x$, $\sin^{-1}(\sin x) = x$

$\cos(\cos^{-1} x) = x$, $\cos^{-1}(\cos x) = x$

$\tan(\tan^{-1} x) = x$, $\tan^{-1}(\tan x) = x$

$\sec(\sec^{-1} x) = x$, $\sec^{-1}(\sec x) = x$

$\csc(\csc^{-1} x) = x$, $\csc^{-1}(\csc x) = x$

$\cot(\cot^{-1} x) = x$, $\cot^{-1}(\cot x) = x$

---

# Derivatives of Inverse Trig Functions

By differentiating the first Cancellation Law for each trig function, and using trigonometric identities we get a differentiation rule for its inverse:

For example:

$\dfrac{d\left(\sin\left(\sin^{-1} x\right)\right)}{dx} = \dfrac{d(x)}{dx}$ so:

$\cos\left(\sin^{-1}x\right)\dfrac{d\left(\sin^{-1}x\right)}{dx} = 1$ and therefore

$$\frac{d\left(\sin^{-1}x\right)}{dx} = \frac{1}{\cos\left(\sin^{-1}x\right)} = \frac{1}{\sqrt{1-x^2}}$$

(Remember that $\cos\left(\sin^{-1}x\right) = \sqrt{1-\left(\sin\left(\sin^{-1}x\right)\right)^2} = \sqrt{1-x^2}$)

---

We list the standard differentiation rules for the six inverse trig functions. The first three should be memorized, and the student should practice deriving them all from first principles as done above.

$$\frac{d\left(\sin^{-1}x\right)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\left(\cos^{-1}x\right)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\left(\tan^{-1}x\right)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d\left(\sec^{-1}x\right)}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d\left(\csc^{-1}x\right)}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d\left(\cot^{-1}x\right)}{dx} = -\frac{1}{1+x^2}$$

---

Remember that these formulas are only valid when the domains are as in the definition of the inverse.

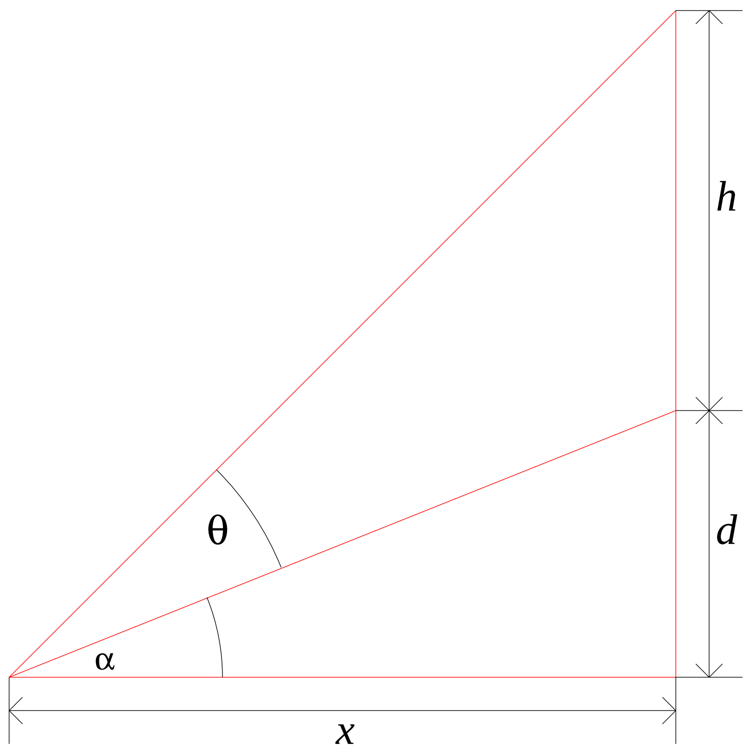
The differentiation rules for the inverse trig functions give us a whole new class of integration formulas, which need not be memorized, because we will soon get into the technique of trigonometric substitution which can be used to easily derive these formulas:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C = -\cos^{-1}x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + C = \cot^{-1}x + C$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C = -\csc x + C =$$

**Example:**Problem 6.7-66(p.412 of the Brown *Stewart*
A painting in an art gallery has height $h$ and is hung so that its lower edge is a distance $d$ above the eye of an observer. How far from the wall should the observer stand so as to maximize the angle $\theta$ subtended at his eye by the painting?




Solution 1: (Using Inverse Trig Functions)

**Variables:**
$x$ = observer's distance from the wall
$\alpha$ = angle between the horizontal and the bottom of the painting
$\theta$ = angle between the top and bottom of the painting

**Relations:**

$$\alpha = \arctan \frac{d}{x}$$

$$\theta + \alpha = \arctan \frac{d+h}{x}$$

$$\theta(x) = \arctan \frac{d+h}{x} - \arctan \frac{d}{x}$$

We have to find the value of $x$ that will make $\theta$ as large as possible, so we differentiate:

$$\theta'(x) = \frac{1}{1+\left(\frac{d+h}{x}\right)^2}\left(-\frac{d+h}{x^2}\right) - \frac{1}{1+\left(\frac{d}{x}\right)^2}\left(-\frac{d}{x^2}\right) =$$

$$-\frac{d+h}{x^2+(d+h)^2}+\frac{d}{x^2+d^2}=-\frac{-(d+h)(x^2+d^2)+d\left(x^2+(d+h)^2\right)}{\left(x^2+(d+h)^2\right)(x^2+d^2)}=$$

$$\frac{-dx^2-d^3-hx^2-hd^2+dx^2+d^3+2d^2h+dh^2}{\left(x^2+(d+h)^2\right)(x^2+d^2)}=$$

$$\frac{h(d^2+dh-x^2)}{\left(x^2+(d+h)^2\right)(x^2+d^2)}=0 \text{ if } x=\sqrt{d(d+h)}$$

Solution 2: (Not Using Inverse Trig Functions)

We use the same variables, but different relations:

**Relations:**

$\tan\alpha=\frac{d}{x}$

$\tan(\theta+\alpha)=\frac{d+h}{x}$

Then we have

$$\tan(\theta+\alpha)=\frac{\tan\alpha+\tan\theta}{1-\tan\alpha\tan\theta}=\frac{\frac{d}{x}+\tan\theta}{1-\frac{d}{x}\tan\theta}=\frac{d+x\tan\theta}{x-d\tan\theta}=\frac{h+d}{x} \text{ or}$$

$x(d+x\tan\theta)=(h+d)(x-d\tan\theta)$ or

$xd+x^2\tan\theta=(h+d)x-(h+d)d\tan\theta$ or

$x^2\tan\theta+(h+d)d\tan\theta=(h+d)x-xd$ or

$\tan\theta\left[x^2+(h+d)d\right]=xh$ or

$$\tan\theta=h\frac{x}{x^2+(h+d)d}$$

We must find the value of $x$ which will make $\tan\theta$ a maximum. Let $f(x)=\frac{x}{x^2+(h+d)d}$, so that $\tan\theta=hf(x)$

Then

$$f'(x)=\left(\frac{x}{x^2+(h+d)d}\right)'=\frac{(x^2+(h+d)d)(x)'-x(x^2+(h+d)d)'}{(x^2+(h+d)d)^2}=$$

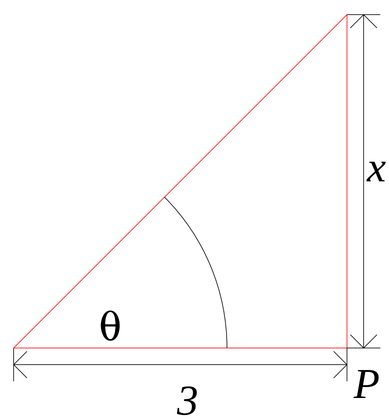
$$\frac{(x^2+(h+d)d)-x(2x)}{(x^2+(h+d)d)^2}=\frac{(h+d)d-x^2}{(x^2+(h+d)d)^2}=0 \text{ if}$$

$(h+d)d-x^2=0$ or $x=\sqrt{(h+d)d}$

This value is known as the *geometric mean of $d$ and $h+d$*.

---

**Example:**Problem 6.7-68(p.412 of the Brown *Stewart*, or 3.6-66(p.234) of the blue *Stewart*
A lighthouse is on a small island 3 km away from the nearest point **P** on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from **P**?




Solution 1: (Using Inverse Trig Functions)

**Variables:**
$x$ = beam's distance from **P**
$\theta$ = angle between beam of light and line through the lighthouse and **P**

**Relations:**
$\theta = \arctan \frac{x}{3}$

Differentiating, we get

$$\theta' = \frac{1}{1+\left(\frac{x}{3}\right)^2}\frac{x'}{3} = \frac{3}{9+x^2}x', \text{ so}$$

$$x' = \frac{9+x^2}{3}\theta' = \frac{9+1^2}{3}\theta' = \frac{10}{3}\theta' = \frac{10}{3}8\pi = \frac{80}{3}\pi.$$

Solution 2: (Not Using Inverse Trig Functions)

We use the relation:

$\tan\theta(t) = \frac{x(t)}{3}$, so differentiation gives
$\sec^2\theta(t)\theta'(t) = \frac{x'(t)}{3}$.

We have $\theta'(t) = 4(2\pi)\frac{\text{radians}}{\text{min}} = 8\pi\frac{\text{radians}}{\text{min}}$, so
$x'(t) = 3\sec^2\theta(t)\theta'(t) = 24\pi\sec^2\theta(t)\frac{\text{km}}{\text{min}}$.
When $x = 1$, $\tan\theta(t) = \frac{1}{3}$, and since $\sec^2\alpha \equiv \tan^2\alpha + 1$, we have $\sec^2\theta(t) = \frac{1}{9} + 1 = \frac{10}{9}$, so
$x'(t) = 24\pi\frac{10}{9}\frac{\text{km}}{\text{min}} = \frac{80\pi}{3}\frac{\text{km}}{\text{min}} = 1600\pi\frac{\text{km}}{\text{hour}}$.

# Indeterminate Forms

**L'Hospital's Rule (1)** If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then, if $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, it must equal $\lim_{x\to a} \frac{f'(x)}{g'(x)}$

**L'Hospital's Rule (2)** If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then, if $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, it must equal $\lim_{x\to a} \frac{f'(x)}{g'(x)}$

---

**Example:** $\lim_{x\to 0} \frac{b^x - 1}{x} = \lim_{x\to 0} \frac{\left(e^{\ln b}\right)^x - 1}{x} = \lim_{x\to 0} \frac{e^{x\ln b} - 1}{x} =$

$\lim_{x\to 0} \frac{\left(e^{x\ln b} - 1\right)'}{(x)'} = \lim_{x\to 0} \frac{\ln b e^{x\ln b}}{1} = \ln b e^{0\ln b} = \ln b$

---

**Example:** If $n$ is a positive integer,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{(e^x)'}{(x^n)'} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$$

---

**Example:** $\lim_{x\to\infty} \frac{\ln x}{\sqrt[n]{x}} = \lim_{x\to\infty} \frac{(\ln x)'}{\left(x^{\frac{1}{n}}\right)'} = \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{n}x^{\frac{1-n}{n}}} = n \lim_{x\to\infty} x^{-1-\frac{1-n}{n}} =$

$\lim_{x\to\infty} n \lim_{x\to\infty} x^{-\frac{1}{n}} = n \lim_{x\to\infty} \frac{1}{x^{\frac{1}{n}}} = 0$

---

**Indeterminate Products:**

The limit of a product where one factor approaches 0 and the appoaches $\pm\infty$ can be computed by conversion to a quotient and applying L'Hospital's Rule:

**Example:** $\lim_{x\to 0^+} xe^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\left(e^{\frac{1}{x}}\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0^+} \frac{-\frac{1}{x^2}e^{\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$

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**Example:** $\lim_{x\to 0^-} xe^{\frac{1}{x}} = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \lim_{x\to 0^-} \frac{\left(e^{\frac{1}{x}}\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0^-} \frac{-\frac{1}{x^2}e^{\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to 0^-} e^{\frac{1}{x}} = 0$

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**Example:**

$$\lim_{x\to 0^-} x\ln x = \lim_{x\to 0^-} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^-} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0^-} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^-} -x = 0$$

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**Indeterminate Differences:**

The limit of a difference where both factors approach $\pm\infty$ can be computed by conversion to a quotient and applying L'Hospital's Rule:

**Example:** $\lim_{x\to 0}(\csc x - \cot x) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) =$

$$\lim_{x\to 0}\left(\frac{(1-\cos x)'}{(\sin x)'}\right) = \lim_{x\to 0}\left(\frac{-(-\sin x)}{\cos x}\right) = \lim_{x\to 0}\tan x = 0$$

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**Indeterminate Powers:**

The limit of a power such as $\lim_{x\to a} f(x)^{g(x)}$ can be computed by conversion to a base $e$ power of a product and applying the techniques just introduced:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a}\left(e^{\ln(f(x))}\right)^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))} = e^{\lim_{x\to a} g(x)\ln(f(x)}$$

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**Example:** $\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} x\ln x} = e^0 = 1$

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**Example:#68,p.428**

$$\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

This is most easily handled by looking at the logarithm of the function,

$$\ln\left(\frac{2x-3}{2x+5}\right)^{2x+1} = (2x+1)\ln\left(\frac{2x-3}{2x+5}\right) = (2x+1)\left(\ln(2x-3)-\ln(2x+5)\right) = \frac{\ln(2x-3)-\ln(2x+5)}{\frac{1}{2x+1}}$$

and applying L'Hospital's Rule to it:

$$\lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{\frac{1}{2x+1}} = \lim_{x\to\infty}\frac{\left(\ln(2x-3)-\ln(2x+5)\right)'}{\left(\frac{1}{2x+1}\right)'} = \lim_{x\to\infty}\frac{\frac{2}{2x-3}-\frac{2}{2x+5}}{\frac{-2}{(2x+1)^2}} =$$

$$\lim_{x\to\infty}\frac{\left(\frac{1}{2x-3}-\frac{1}{2x+5}\right)}{\frac{-1}{(2x+1)^2}} = -\lim_{x\to\infty}(2x+1)^2\left(\frac{1}{2x-3}-\frac{1}{2x+5}\right) =$$

$$-\lim_{x\to\infty}(2x+1)^2\left(\frac{2x+5}{(2x-3)(2x+5)}-\frac{2x-3}{(2x-3)(2x+5)}\right) = -\lim_{x\to\infty}(2x+1)^2\left(\frac{8}{(2x-3)(2x+5)}\right) =$$

$$-8\lim_{x\to\infty}\left(\frac{(2x+1)^2}{(2x-3)(2x+5)}\right) = -8$$

so the desired limit is $e^{-8}$