Generalized Logarithms and Exponentials

If $a > 0$, we define, for any real number $a > 0$, the Generalized Exponential Functions

$$f(x) = a^x := e^{(\ln a)x} > 0.$$

These functions have two important classes of properties, which are easily derived from those of e^x:

Arithmetic Properties:

$a^x a^y = a^{x+y}$, since $a^x a^y = e^{(\ln a)x} e^{(\ln a)y} = e^{(\ln a)(x+y)} = a^{x+y}$

$\dfrac{a^x}{a^y} = a^{x-y}$, since $\dfrac{a^x}{a^y} = \dfrac{e^{(\ln a)x}}{e^{(\ln a)y}} = e^{(\ln a)(x-y)} = a^{x-y}$

$(a^x)^y = a^{xy}$, since $(a^x)^y = \left(e^{(\ln a)x}\right)^y = \left(e^{((\ln a)x)y}\right) = a^{xy}$

Calculus Properties:

First Derivative:

$(a^x)' = \left(e^{(\ln a)x}\right)' = e^{(\ln a)x}((\ln a)x)' = e^{\ln ax}(\ln a) = (\ln a)e^{\ln ax} = (\ln a)a^x$

or

$$(a^x)' = (\ln a)a^x$$

Antiderivative: $\int a^x dx = \frac{1}{\ln a}a^x + C$

Second Derivative:

$$(a^x)'' = (\ln a)^2 a^x \geq 0.$$

Thus the second derivative is positive if $a \neq 1$, and the first derivative is positive is $a > 1$ and is negative if $0 < a < 1$, so we have two possible types of graph of $y = a^x$. These are best viewed with a Java applet .

The Inverse of a^x

In either case, so long as $a > 0$ and $a \neq 1$, we have that $f(x) = a^x$ is one-to-one, with domain $(-\infty, \infty)$, and range $(0, \infty)$, so $f(x) = a^x$ has an inverse f^{inv} with domain $(0, \infty)$ and range $(-\infty, \infty)$.

This inverse is usually written $\log_a x$, and is called the **logarithm to base a of x** .

The Cancellation Equations for this inverse are:

$$a^{\log_a x} = x \quad \text{and} \quad \log_a (a^x) = x$$

Fron the defintion of $a^x = e^{(\ln a x)}$ we have $a^{\log_a x} = e^{(}\ln a \log_a x) = x$,

and since e^x is one-to-one and $e^{\ln x} = x$, we must have $\ln a \log_a x = \ln x$, so we get a very useful formula for $\log_a x$:

$$\log_a x = \frac{\ln x}{\ln a}$$

The logarithm functions have two important classes of properties:

Arithmetic Properties:

$\log_a xy = \log_a x + \log_a y$, since

$$\log_a xy = \frac{\ln xy}{\ln a} = \frac{\ln x + \ln y}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} = \log_a x + \log_a y$$

$\log_a \frac{x}{y} = \log_a x - \log_a y$, since

$$\log_a \frac{x}{y} = \frac{\ln \frac{x}{y}}{\ln a} = \frac{\ln x - \ln y}{\ln a} = \frac{\ln x}{\ln a} - \frac{\ln y}{\ln a} = \log_a x - \log_a y$$

$\log_a x^y = y \log_a x$, since

$$\log_a x^y = \frac{\ln x^y}{\ln a} = \frac{y \ln x}{\ln a} = y \frac{\ln x}{\ln a} = y \log_a x$$

Calculus Properties:

First Derivative:

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x \ln a}$$

or

$$(\log_a x)' = \frac{1}{x \ln a}$$

Second Derivative:

$$(\log_a x)'' = -\frac{1}{x^2 \ln a}$$

Thus the signs of the first and second derivatives depend only on the sign of $\ln a$. The graphs of the logarithm functions are best viewed with a Java applet .

Uses of “Non-Natural” Logarithms

The logarithm to base 10 is called the **common logarithm**. Until about 1970, it was essential for all technically skilled persons to be extremely dextrous in the use of common logarithms because without easily available electronic computing devices common logarithms were the method of choice to be used in technical computations.

Since the advent of microchip technology this dexterity is no longer needed on a daily basis by technical personnel. The natural logarithm is now preferred in the remaining extremely important use of exponential and logarithm functions: the modelling of processes characterized by **constant relative rates of change** .

There are, however, some remnants of the common logarithm still in current use: scientific measurements of magnitudes of earthquakes, sound levels, pH or acidity levels are expressed in units based on logarithms to the base 10.

Logarithms to base 2 are also frequently used in computational science in the analysis of complexity of calculations.