The Exponential Function

Since $F(x) = \ln x$ is one-to-one on $(0, \infty)$ and has range $(-\infty, \infty)$, it has an inverse function F^{inv} which is called the **exponential function** which has domain $(-\infty, \infty)$ and range $(0, \infty)$.

It is usually written as e^x, and the Cancellation Laws give us

$e^{\ln x} = x$ and $\ln(e^x) = x$.

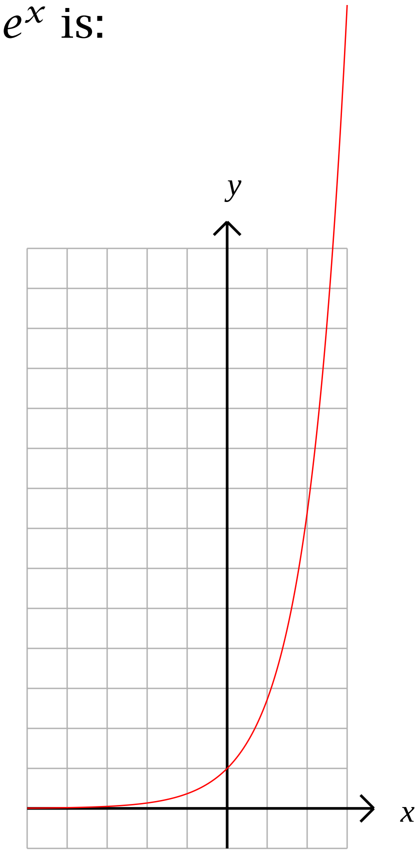
The exponential function inherits its important arithmetic properties from $\ln x$:

$e^{a+b} = e^a e^b$, $e^{a-b} = \dfrac{e^a}{e^b}$, and $(e^a)^b = e^{ab}$.

Differentiation of the equation $\ln(e^x) = x$ gives us $\dfrac{1}{e^x}\dfrac{d}{dx}(e^x) = 1$, so we have

$\dfrac{d}{dx}(e^x) = e^x > 0$, so e^x is one of the very special functions which equals its own derivative, which is always positive.

The graph of $y = e^x$ is:



Clearly $e^0 = 1$. The value of e^1 is called e, and equals, to 20 decimal places,

2.71828182845904523536.

We have the important differentiation formula:

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$$

Example: Sketch the graph of $y = f(x) = xe^{-x^2}$

Solution: We have

$$y' = x\left(e^{-x^2}\right)' + (x)'e^{-x^2} = x\left(e^{-x^2}(-2x)\right) + (1)e^{-x^2} = (1-2x^2)e^{-x^2}$$

and

$$y'' = (1-2x^2)\left(e^{-x^2}\right)' + (1-2x^2)'e^{-x^2} =$$

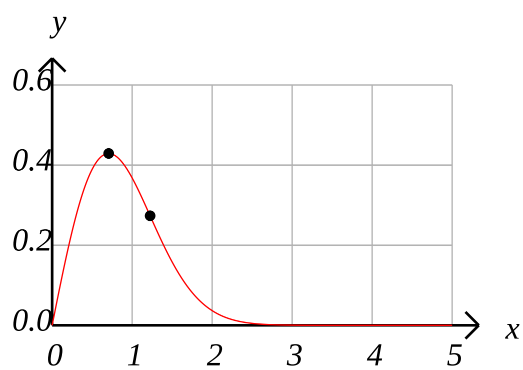
$$(1-2x^2)\left(e^{-x^2}(-2x)\right) + (-4x)e^{-x^2} = -2x\left(3-2x^2\right)e^{-x^2}$$

The critical numbers of f are $-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$, and the inflection numbers are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$.

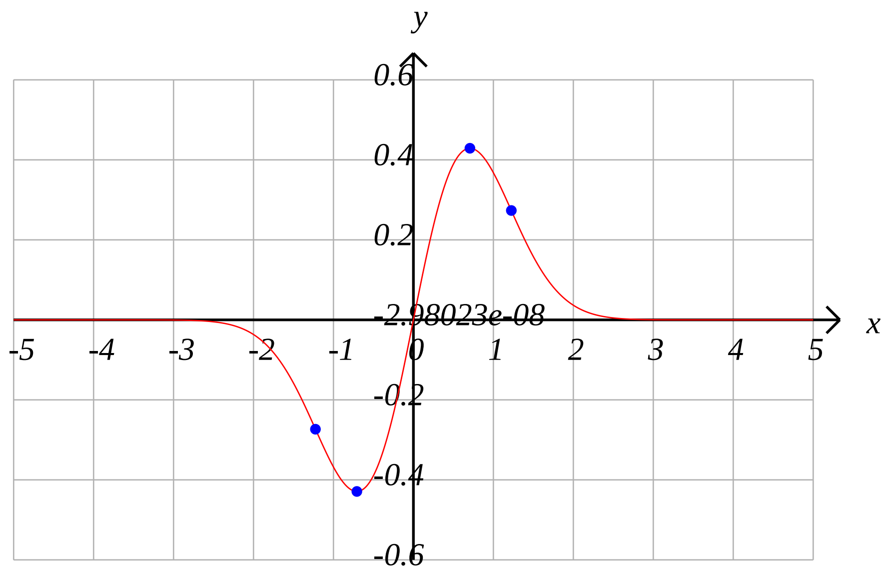
Since f is an odd function, we only look at a table of signs for non-negative x:

x	0	$\left(0, \frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}\right)$	$\sqrt{\frac{3}{2}}$	$\left(\sqrt{\frac{3}{2}}, \infty\right)$	∞
$f''(x)$	0	$-$	$-$	$-$	0	$+$	$+\infty$
$f'(x)$	1	$+$	0	$-$	$-$	$-$	0
$f(x)$	0	$+$	$+$	$+$	$+$	$+$	0

We use this to sketch a graph of f for non-negative x:



and then we use this to sketch the whole graph:


y
x
0.6
0.4
0.2
-2.98023e-08
-0.2
-0.4
-0.6
-5
-4
-3
-2
-1
0
1
2
3
4
5

We also have the integration formula

$$\int e^x = e^x + C$$