

Exponential Growth and Decay

Recall that if $y = f(t)$, then $f'(t) = \frac{dy}{dt}$ is called the **rate of change** of y with respect to t .

Another very important measure of rate of change is the **relative rate of change**

$$\frac{f'(t)}{f(t)} = \frac{y'}{y}.$$

If this quantity is a constant, say k , then there is a nice formula for y as a function of t :

$y = y_0 e^{k(t-t_0)}$, where y_0 is the value of y when $t = t_0$.

If $t_0 = 0$, the formula becomes even nicer: $y = y_0 e^{kt}$




If $k > 0$, so that y is increasing with t , we say that we have **exponential growth** .

If $k < 0$, so that y is decreasing with t , we say that we have **exponential decay** .

It is usually clear from the context of a situation that the relative rate of change is a constant.

The practical problem is to translate given data into a useful equation for the quantity that is of interest.



Some Real World Models:

(1) Model #1: Growth of Bacterial Cultures

It is known that under certain conditions a population of bacteria grow at a constant relative rate.

The standard equation to represent this is

$P(t) = P(0)e^{kt}$, where $P(0)$ is the population at time 0, and k is a constant that depends on the type of bacteria and its environmental conditions.

Usually, this problem arises in a context where the population value is measured at two different points in time, and it is required to use this information to predict the population values at other times.

In other words, two **Data Points** (t_1, P_1) and (t_2, P_2) are given, and one wishes to find the curve $P(t) = P(0)e^{kt}$ that passes through them.

The simplest of such problems are those where the initial value of the population is given.

Example: A bacteria culture starts with 4000 bacteria, and in 3 hours has grown to 7000.

- (1) Find a formula for the amount of bacteria after t hours.
- (2) How long will it take the population to double?
- (3) How long will it take the population to grow by a factor of 10?

Solution: We have $P(0) = 4000$, and $P(3) = 7000$.

We know that the population function satisfies the equation $P(t) = P(0)e^{kt}$, and since we have $P(0) = 4000$, we have $P(t) = 4000e^{kt}$.

We need to find k , so we use the other Data Point:

$$P(3) = 7000 = 4000e^{k(3)} = 4000e^{3k}.$$

We must solve for k :

$$\text{We have } 7000 = 4000e^{3k}, \text{ so } \frac{7000}{4000} = e^{3k} \text{ or } \frac{7}{4} = e^{3k}$$

Taking natural logarithms, we have:

$$\ln \frac{7}{4} = \ln e^{3k} = 3k, \text{ so } k = \frac{\ln \frac{7}{4}}{3}$$

and thus

$$P(t) = 4000e^{kt} = 4000e^{\frac{\ln \frac{7}{4}}{3}t} = 4000 \left(\frac{7}{4}\right)^{\frac{t}{3}}$$

so now we have the desired formula for $P(t)$.

There are still two unanswered questions:

(2) How long will it take the population to double?

Since $P(0) = 4000$, we need to find the value of t for which $P(t) = 8000$, so we solve the equation:

$$8000 = 4000e^{\frac{\ln \frac{7}{4}}{3}t} \text{ for } t:$$

$$2 = e^{\frac{\ln \frac{7}{4}}{3}t}$$

$$\ln 2 = \frac{\ln \frac{7}{4}}{3} t$$

$$3 \ln 2 = \left(\ln \frac{7}{4} \right) t$$

$$t = \frac{3 \ln 2}{\ln \frac{7}{4}} = \frac{3 \ln 2}{\ln 7 - \ln 4} \doteq 3.71$$

This number is called the **doubling time** of an exponential growth model, and is denoted by \mathcal{T}_2 .

For a growth equation $y = y_0 e^{kt}$, we have $\mathcal{T}_2 = \frac{\ln 2}{k}$.

(3) How long will it take the population to grow by a factor of 10?

Since $P(0) = 4000$, we need to find the value of t for which $P(t) = 40000$, so we solve the equation:

$$10000 = 4000e^{\frac{\ln 7}{3}t} \text{ for } t:$$

$$10 = e^{\frac{\ln 7}{3}t}$$

$$\ln 10 = \frac{\ln 7}{3}t$$

$$3 \ln 10 = \left(\ln \frac{7}{4}\right) t$$

$$t = \frac{3 \ln 10}{\ln \frac{7}{4}} = \frac{3 \ln 10}{\ln 7 - \ln 4}$$

This number is called the **magnitude growth time** of an exponential growth model, and is denoted by \mathcal{T}_{10} .

For a growth equation $y = y_0 e^{kt}$, we have $\mathcal{T}_{10} = \frac{\ln 10}{k}$.

Model #2: Radioactive Decay

It is known that radioactive elements decay at a rate proportional to the amount of the element present: in other words, the relative rate of change is constant, and negative.

For example, radium-226 decays in such a way that half of it disappears every 1590 years. The decay equation

$y(t) = y_0 e^{kt}$ must therefore satisfy $\frac{1}{2}y_0 = y_0 e^{k(1590)}$ so we can solve for k by eliminating y_0 and taking logarithms:

$$\frac{1}{2} = e^{1590k}$$

$$\ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2 = 1590k$$

so

$$k = -\frac{\ln 2}{1590} \doteq -0.0004359$$

Many people prefer to have the constant k positive, so they would start with the decay equation in the form

$$y(t) = y_0 e^{-kt}.$$

In either case, the resulting model equation is

$$y(t) = y_0 e^{-\frac{\ln 2}{1590}t} = y_0 2^{-\frac{t}{1590}}$$

Note that it is easier to understand the model's equation when it is written in terms of powers of 2, rather than powers of e . It is essential that the student be able to move easily from one form to another. The number 1590 is called the **half-life** of the decaying element, and is denoted by $\mathcal{T}_{\frac{1}{2}}$ or $\mathcal{T}_{0.5}$.

If the decay equation is $y(t) = y_0 e^{-kt}$, ($k > 0$), then we have $\mathcal{T}_{\frac{1}{2}} = \frac{\ln 2}{k}$.

The length of time it takes an element to decay to one-tenth of its original value is called the **magnitude decay time**, denoted by $\mathcal{T}_{\frac{1}{10}}$ or $\mathcal{T}_{0.1}$ and

is related to $k (> 0)$ by $\mathcal{T}_{\frac{1}{10}} = \frac{\ln 10}{k}$.

The General Problem

Suppose we are told that a quantity y varies with time t at a constant relative rate of change, and we are given two data points (t_1, y_1) and (t_2, y_2) , from which we are to construct a model growth or decay equation which is then to be used to determine the doubling time or half-life and the magnitude growth or decay time.

Since we have two readings, we can insert them into the equation

$$y(t) = y_0 e^{kt}$$

and get two data equations in the two unknown constants y_0 and k :

$$y_1 = y_0 e^{kt_1} \text{ and } y_2 = y_0 e^{kt_2}.$$

Taking the ratios of the two equations, we get

$$\frac{y_2}{y_1} = \frac{y_0 e^{kt_2}}{y_0 e^{kt_1}} = e^{k(t_2 - t_1)}$$

and on taking logarithms we have

$$\ln \left(\frac{y_2}{y_1} \right) = \ln y_2 - \ln y_1 = k(t_2 - t_1) \text{ so}$$

$$k = \frac{\ln y_2 - \ln y_1}{t_2 - t_1}$$

We can now use this in the first data equation to find the remaining unknown constant y_0 :

$$y_1 = y_0 e^{kt_1} = y_0 e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1}, \text{ so}$$

$$y_0 = y_1 e^{-\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1}$$

If we now put these values in the model equation,

$y(t) = y_0 e^{kt}$, we get

$$y(t) = y_1 e^{-\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1} e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t}$$

or

$$y(t) = y_1 e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1} (t - t_1)} = y_1 \left(\frac{y_2}{y_1} \right)^{\frac{t - t_1}{t_2 - t_1}}$$

We could just as easily derive

$$y(t) = y_2 \left(\frac{y_1}{y_2} \right)^{\frac{t - t_2}{t_1 - t_2}}$$


The advantage of these two formulas is that is easy to check that they pass through the given data points.

Of course, when doing practical calculations, these equation should, and often must, be converted to base e and natural logarithm format.

Model#3 Newton's Law of Temperature Change

Newton's Law of Temperature Change says that the rate of change of the temperature of an object is directly proportional to the difference between its temperature, and the temperature of its surroundings, called the **ambient temperature**.

Industrial applications of this model are plentiful: any situation which requires the heating or cooling of significant quantities of materials is modeled by this law. Examples are found in steel mills, large bakeries, chemical complexes, especially breweries, etc. It is also used by metallurgical engineers in the tempering of metal products.



The Model

We will let the temperature of the object be $T(t)$, and since this will tend to the ambient temperature as $t \rightarrow \infty$, we will denote the ambient temperature by T_∞ .

We will also denote the initial temperature $T(0)$ by T_0 .

It is important to notice that the temperature $T(t)$ does **not** have a constant proportional rate of change:

it is the function $\mathcal{D}(t) = T(t) - T_\infty$ that satisfies

$$\frac{\mathcal{D}'(t)}{\mathcal{D}(t)} = -k \text{ for some constant } k > 0.$$

We therefore have $\mathcal{D}(t) = \mathcal{D}(0)e^{-kt} = (T_0 - T_\infty)e^{-kt}$, or, converting to expressions involving $T(t)$ only:

$$T(t) - T_\infty = (T_0 - T_\infty)e^{-kt}$$


or

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-kt} = T_\infty(1 - e^{-kt}) + T_0e^{-kt}$$

The Real World

If we are lucky, we will know the initial temperature T_0 and the ambient temperature T_∞ .

We may, for example, want to cook a turkey, which we have brought home from the butcher thawed and at a temperature of 5°C . Leaving it in the kitchen sink, where the temperature is 20°C , while we prepare the stuffing, we observe that its temperature increases by 1°C in the 30 minutes it takes us to prepare the stuffing and to stuff the turkey. Our cookbook tells us to cook the turkey at an oven temperature of 180°C . If the turkey is considered to be cooked when it has reached the temperature of 85°C , how long will it take to cook the turkey?



Solution:

Phase I:Pre-Oven

Before the turkey is in the oven, we have $T_0 = 5$ and $T_\infty = 20$, and we also have $T(30) = 6$, so we can solve for k :

$$T(30) = T_\infty (1 - e^{-k(30)}) + T_0 e^{-k(30)} = 6 = 20 (1 - e^{-k(30)}) + 5e^{-k(30)}$$

$$6 = 20 - 20e^{-30k} + 5e^{-30k} = 20 - 15e^{-30k}$$

$$-14 = -15e^{-30k}$$

$$e^{-30k} = \frac{-14}{-15}$$

$$-30k = \ln\left(\frac{14}{15}\right)$$

$$k = \frac{\ln 14 - \ln 15}{-30} = \frac{\ln 15 - \ln 14}{30} \doteq 0.0022997$$

Phase II: In the Oven

When we put the turkey in the oven, we have $T_0 = 6$, and $T_\infty = 180$, so the temperature at time t will be

$$T(t) = T_\infty (1 - e^{-kt}) + T_0 e^{-kt} = 180 \left(1 - e^{-\frac{\ln 15 - \ln 14}{30}t}\right) + 6e^{-\frac{\ln 15 - \ln 14}{30}t} =$$
$$180 - 174e^{-\frac{\ln 15 - \ln 14}{30}t}$$

We have to find the time t when this equals 85:

$$85 = 180 - 174e^{-\frac{\ln 15 - \ln 14}{30}t} \text{ if}$$

$$-95 = -174e^{-\frac{\ln 15 - \ln 14}{30}t} \text{ if}$$

$$\frac{-95}{-174} = e^{-\frac{\ln 15 - \ln 14}{30}t} \text{ if (taking logarithms)}$$

$$\ln\left(\frac{95}{174}\right) = -\frac{\ln 15 - \ln 14}{30}t \text{ if}$$

$$\ln 95 - \ln 174 = -\frac{\ln 15 - \ln 14}{30}t \text{ if}$$

$$30 \frac{\ln 95 - \ln 174}{-(\ln 15 - \ln 14)} = t \text{ if}$$

$$t = 30 \frac{\ln 174 - \ln 95}{\ln 15 - \ln 14} \doteq 30 \frac{0.6051784}{0.0689929} \doteq 263 \text{ minutes}$$

or about 4.4 hours.

Three Temperature Readings

It sometimes happens that three temperature readings (t_1, T_1) , (t_2, T_2) , and (t_3, T_3) are taken, and it is desired to use them to find the complete equation of the temperature of the object being studied.

We get three equations in the three unknown constants T_0 , T_∞ , and k :

$$T_1 = T_\infty + (T_0 - T_\infty) e^{-kt_1}$$

$$T_2 = T_\infty + (T_0 - T_\infty) e^{-kt_2}$$

$$T_3 = T_\infty + (T_0 - T_\infty) e^{-kt_3}$$

Without some extra information these equations are usually impossible to solve. It is usually easy to arrange for the times between the three measurements to be equal, say to Δt . Then we have $t_2 = t_1 + \Delta t$ and $t_3 = t_1 + 2\Delta t$, and it is possible to eliminate T_0 and k :

$$T_1 - T_\infty = (T_0 - T_\infty) e^{-kt_1}$$

$$T_2 - T_\infty = (T_0 - T_\infty) e^{-k(t_1 + \Delta t)}$$

$$T_3 - T_\infty = (T_0 - T_\infty) e^{-k(t_1 + 2\Delta t)}$$

$$(A) \frac{T_1 - T_\infty}{T_0 - T_\infty} = e^{-kt_1}$$

$$(B) \frac{T_2 - T_\infty}{T_0 - T_\infty} = e^{-k(t_1 + \Delta t)}$$

$$(C) \frac{T_3 - T_\infty}{T_0 - T_\infty} = e^{-k(t_1 + 2\Delta t)}$$

Now we divide equation (B) by equation (A) and equation (C) by equation (A):

$$\frac{T_2 - T_\infty}{T_1 - T_\infty} = \frac{e^{-k(t_1 + \Delta t)}}{e^{-kt_1}} = e^{-k\Delta t}$$

$$\frac{T_3 - T_\infty}{T_2 - T_\infty} = \frac{e^{-k(t_1 + 2\Delta t)}}{e^{-k(t_1 + \Delta t)}} = e^{-k\Delta t}$$

so we have

$$\frac{T_2 - T_\infty}{T_1 - T_\infty} = \frac{T_3 - T_\infty}{T_2 - T_\infty}$$

Note that T_0 has disappeared. Simplifying, we get:

$$(T_2 - T_\infty)(T_2 - T_\infty) = (T_1 - T_\infty)(T_3 - T_\infty)$$

$$\text{or } T_2^2 - 2T_2T_\infty + T_\infty^2 = T_1T_3 - (T_1 + T_3)T_\infty + T_\infty^2$$

$$\text{or } (T_1 - 2T_2 + T_3)T_\infty = T_1T_3 - T_2^2$$

$$\text{or } T_\infty = \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3}$$

We now use this to find:

$$\begin{aligned}
 e^{-k\Delta t} &= \frac{T_2 - \frac{T_1 T_3 - T_2^2}{T_1 - 2T_2 + T_3}}{T_1 - \frac{T_1 T_3 - T_2^2}{T_1 - 2T_2 + T_3}} = \\
 &= \frac{T_2(T_1 - 2T_2 + T_3) - (T_1 T_3 - T_2^2)}{T_1(T_1 - 2T_2 + T_3) - (T_1 T_3 - T_2^2)} = \\
 &= \frac{T_1 T_2 - 2T_2^2 + T_2 T_3 - T_1 T_3 + T_2^2}{T_1^2 - 2T_1 T_2 + T_1 T_3 - T_1 T_3 + T_2^2} = \\
 &= \frac{T_1 T_2 - T_2^2 + T_2 T_3 - T_1 T_3}{T_1^2 - 2T_1 T_2 + T_2^2} = \\
 &= \frac{T_2(T_1 - T_2) + T_3(T_2 - T_1)}{(T_1 - T_2)^2} = \\
 &= \frac{(T_3 - T_2)(T_2 - T_1)}{(T_2 - T_1)^2} = \frac{T_3 - T_2}{T_2 - T_1}
 \end{aligned}$$

If this quantity is not positive there is no mathematical solution, which should tell us that there is a problem with the data or with the model. If this quantity is positive but is greater than 1 there is also a problem, because this would tell us that the temperature is going to tend to $\pm\infty$!

Using $e^{-k\Delta t} = \frac{T_3 - T_2}{T_2 - T_1}$ we can solve explicitly for k :

$$-k\Delta t = \ln\left(\frac{T_3 - T_2}{T_2 - T_1}\right) = \ln(T_3 - T_2) - \ln(T_2 - T_1)$$

$$k = \frac{\ln(T_2 - T_1) - \ln(T_3 - T_2)}{\Delta t}$$

Next we substitute the values of k and T_∞ into equation (A) to find T_0 :

(A) $\frac{T_1 - T_\infty}{T_0 - T_\infty} = e^{-kt_1}$ becomes

$$\frac{T_1 - \frac{T_1 T_3 - T_2^2}{T_1 - 2T_2 + T_3}}{T_0 - \frac{T_1 T_3 - T_2^2}{T_1 - 2T_2 + T_3}} = e^{-\left(\frac{\ln(T_2 - T_1) - \ln(T_3 - T_2)}{\Delta t}\right)t_1} \text{ or}$$

$$\frac{T_1(T_1 - 2T_2 + T_3) - (T_1 T_3 - T_2^2)}{T_0(T_1 - 2T_2 + T_3) - (T_1 T_3 - T_2^2)} = e^{\ln\left(\frac{T_3 - T_2}{T_2 - T_1}\right)\frac{t_1}{\Delta t}} \text{ or}$$

$$\frac{T_1^2 - 2T_1 T_2 + T_1 T_3 - T_1 T_3 + T_2^2}{T_0(T_1 - 2T_2 + T_3) - (T_1 T_3 - T_2^2)} = \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{t_1}{\Delta t}} \text{ or}$$

$$\frac{(T_1 - T_2)^2}{T_0(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2)} = \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{t_1}{\Delta t}} \text{ or}$$

$$(T_1 - T_2)^2 \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{-t_1}{\Delta t}} = T_0(T_1 - 2T_2 + T_3) - (T_1T_3 - T_2^2) \text{ or}$$

$$(T_1 - T_2)^2 \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{-t_1}{\Delta t}} + (T_1T_3 - T_2^2) = T_0(T_1 - 2T_2 + T_3) \text{ or}$$

$$T_0 = \frac{(T_1 - T_2)^2 \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{-t_1}{\Delta t}} + (T_1T_3 - T_2^2)}{T_1 - 2T_2 + T_3}$$

We now insert these values into the formula for $T(t)$:

$$T(t) = T_\infty + (T_0 - T_\infty) e^{-kt}$$

becomes:

$$T(t) = \frac{T_1T_3 - T_2^2}{T_1 - 2T_2 + T_3} + \left(\frac{(T_1 - T_2)^2}{T_1 - 2T_2 + T_3}\right) \left(\frac{T_3 - T_2}{T_2 - T_1}\right)^{\frac{t-t_1}{\Delta t}}$$

It is often convenient to change the notation: we let

$\Delta T_1 = T_2 - T_1$ and $\Delta T_2 = T_3 - T_2$, we get $T_2 = T_1 + \Delta T_1$, and $T_3 = T_1 + \Delta T_1 + \Delta T_2$, so

$$T_\infty = \frac{T_1 T_3 - T_2^2}{T_1 - 2T_2 + T_3} = \frac{T_1(T_1 + \Delta T_1 + \Delta T_2) - (T_1 + \Delta T_1)^2}{\Delta T_2 - \Delta T_1} =$$

$$\frac{T_1^2 + T_1 \Delta T_1 + T_1 \Delta T_2 - (T_1^2 + 2T_1 \Delta T_1 + \Delta T_1^2)}{\Delta T_2 - \Delta T_1} = \frac{T_1(\Delta T_2 - \Delta T_1) - \Delta T_1^2}{\Delta T_2 - \Delta T_1} =$$

$$T_1 - \frac{\Delta T_1^2}{\Delta T_2 - \Delta T_1}$$

$$e^{-k\Delta t} = \frac{T_3 - T_2}{T_2 - T_1} = \frac{\Delta T_2}{\Delta T_1}$$

$$T_0 = T_1 + \frac{(\Delta T_1)^2 \left(\left(\frac{\Delta T_2}{\Delta T_1} \right)^{\frac{-t_1}{\Delta t}} - 1 \right)}{\Delta T_2 - \Delta T_1}$$

$$T(t) = T_1 + \frac{(\Delta T_1)^2}{\Delta T_2 - \Delta T_1} \left[\left(\frac{\Delta T_2}{\Delta T_1} \right)^{\frac{t-t_1}{\Delta t}} - 1 \right]$$

Model # 4 Dilution of Chemicals

Suppose a tank contains a volume V litres of fluid with a concentration c_0 of chemical X, and that there is an inflow into the tank of the fluid at the rate r litres per minute with a concentration level of c_{in} of chemical X. Assuming that the tank is continuously and thoroughly mixed, so that the concentration is uniform throughout the tank, and that fluid is drawn off at the same rate, so that the volume remains constant.

Clearly the concentration will tend to c_{in} , and the amount of chemical X in the tank will therefore tend to $c_{in}V$. Let $x(t)$ be the amount of chemical X, and let $c_{out}(t)$ be the concentration of chemical X in the tank at time t . We have $c_{out}(t) = \frac{x(t)}{V} \frac{\text{kg}}{\ell}$, and

$$\begin{aligned} x'(t) &= \left[c_{in} \frac{\text{kg}}{\ell} - c_{out}(t) \right] r \frac{\ell}{\text{min}} = \left[c_{in} \frac{\text{kg}}{\ell} - \frac{x(t)}{V} \frac{\text{kg}}{\ell} \right] r \frac{\ell}{\text{min}} \\ &= \left[\frac{c_{in} - x(t)}{V} \right] r \frac{\text{kg}}{\text{min}} \end{aligned}$$

Next we let $\mathcal{D}(t) = x(t) - c_{in}V$, so that

$$\mathcal{D}'(t) = x'(t) = \left[\frac{c_{in} - x(t)}{V} \right] r \frac{\text{kg}}{\text{min}} = -\mathcal{D}(t) \frac{r}{V} \frac{\text{kg}}{\text{min}}$$

or, discarding units for the moment,

$$\frac{\mathcal{D}'(t)}{\mathcal{D}(t)} = -\frac{r}{V},$$

which has solution $\mathcal{D}(t) = \mathcal{D}(0)e^{-\frac{r}{V}t}$.

This can now be converted into expressions for $x(t)$ and $c(t)$:

We have $\mathcal{D}(0) = x(0) - c_{in}V = c_0V - c_{in}V = (c_0 - c_{in})V$, so

$\mathcal{D}(t) = (c_0 - c_{in})Ve^{-\frac{r}{V}t}$, and thus

$x(t) - c_{in}V = (c_0 - c_{in})Ve^{-\frac{r}{V}t}$, giving us

$x(t) = c_{in}V + (c_0 - c_{in})Ve^{-\frac{r}{V}t} = [c_{in} + (c_0 - c_{in})e^{-\frac{r}{V}t}]V$ and

$c(t) = c_{in} + (c_0 - c_{in})e^{-\frac{r}{V}t}$.