# Differential Equations

**Definition:** A **differential equation** is any equation containing a (possibly unknown) function and one or more of its derivatives.

**Examples:** $y' = \sin x \qquad y'''' + y' + y = 3 \qquad (y'')^3 + y' = e^{2x}$

The **order** of a differential equation is the order of the highest derivative appearing in it.

We shall usually be looking for functions which satisfy a given differential equation and other natural conditions. These are called **solutions** of the differential equation (d.e. for short). In this course we will only study very simple first order d.e.'s.

# Separable Equations

**Definition:** A first order d.e. is said to be **separable** if it can be written in the form

$$y' = g(x)h(y)$$

Such equations can be manipulated so as to have the variables $x$ and $y$ **separated**:

$\dfrac{dy}{dx} = g(x)h(y)$ is equivalent to

$\dfrac{dy}{h(y)} = g(x)dx,$

and a general solution is given by

$\displaystyle\int \frac{dy}{h(y)} = \int g(x)dx$ and will look like $H(y) = G(x) + C$

**Example:** $y' = \frac{x}{y}$ is separable, because it can be re-written as

$y' = x\frac{1}{y}$ and then can be put into the form

$\frac{dy}{dx} = x\frac{1}{y}$ or $y\,dy = x\,dx$. Integration gives

$\int y\,dy = \int x\,dx$ or $\frac{y^2}{2} = \frac{x^2}{2} + C$

This is equivalent to saying that all solutions of $y' = \frac{x}{y}$ must satisfy $x^2 - y^2 = -2C$ for some constant $C$.

Graphically, we see that the family of hyperbolas $x^2 - y^2 = -2C$, $C \neq 0$, plus the pair of lines $y = x$ and $y = -x$, constitutes the set of solutions of the given differential equation. Specifying a point $(x_0, y_0)$ through which such a hyperbola must pass is equivalent to what is called an **initial condition.**

For example, if the solution is to pass through the point (1,2), we must take $C = \frac{1^2 - 2^2}{-2} = \frac{3}{2}$.

# Constant Relative Rate of Change

The most important differential equation we have so far encountered is $y' - ky = 0$ or $\dfrac{y'}{y} = k$,

that is, situations where the relative rate of change is a constant $k$. This is easily solved by separating variables:

$\dfrac{\frac{dy}{dt}}{y} = k$ becomes $\dfrac{dy}{y} = kdt$, which we integrate:

$\displaystyle\int \frac{dy}{y} = \int kdt$ results in $\ln|y| = kt + C$.

Taking exponentials of both sides, we get

$|y| = e^{kt+C} = e^{C}e^{kt}$ or $y = \pm e^{C}e^{kt}$.

We usually write this in the form $y(t) = y(0)e^{kt}$.

## The Basic Technique: Finding Parameters from Data Points

If we are given that a variable $y$ has constant relative rate of change and two value of the variable at two different times, we can find the parameters $y(0)$ and $k$. Suppose we are given that $y(t) = y_1$ at time $t_1$ and $y(t) = y_2$ at time $t_2$. This means that the graph of $y$ passes through the two points $(t_1, y_1)$ and $(t_2, y_2)$.

Since we know that $y(t) = y(0)e^{kt}$, we get two equations in the two unknown parameters $y(0)$ and $k$:

$y_1 = y(0)e^{kt_1}$ and $y_2 = y(0)e^{kt_2}$. Taking ratios, we get:

$\frac{y_2}{y_1} = \frac{e^{kt_2}}{e^{kt_1}} = e^{k(t_2-t_1)}$, an equation involving only one unknown parameter, $k$. Taking natural logarithms, we get:

$\ln\left(\frac{y_2}{y_1}\right) = \ln\left(e^{k(t_2-t_1)}\right)$ or

$\ln y_2 - \ln y_1 = k(t_2 - t_1)$ or

$$k = \frac{\ln y_2 - \ln y_1}{t_2 - t_1} = \frac{\ln\left(\frac{y_2}{y_1}\right)}{t_2 - t_1}.$$

We can now find $y(0)$ by substituting this value of $k$ into the equation $y_1 = y(0)e^{kt_1}$:

$y_1 = y(0)e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1}t_1}$. Taking logarithms, we get:
$\ln y_1 = \ln y(0) + \dfrac{\ln y_2 - \ln y_1}{t_2 - t_1}t_1$, so
$\ln y(0) = \ln y_1 - \dfrac{\ln y_2 - \ln y_1}{t_2 - t_1}t_1 = \dfrac{(t_2 - t_1)\ln y_1 - t_1 \ln y_2 + t_1 \ln y_1}{t_2 - t_1} =$

$$\frac{t_2 \ln y_1 - t_1 \ln y_2}{t_2 - t_1} = \frac{\ln y_1^{t_2} - \ln y_2^{t_1}}{t_2 - t_1} = \frac{1}{t_2 - t_1} \ln \frac{y_1^{t_2}}{y_2^{t_1}}$$

Therefore $y(0) = \left(\dfrac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2 - t_1}}$ and

$$y(t) = \left(\frac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2-t_1}} e^{\frac{\ln\left(\frac{y_2}{y_1}\right)}{t_2-t_1}t} =$$

$$\left(\frac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2-t_1}} \left(\frac{y_2}{y_1}\right)^{\frac{t}{t_2-t_1}} =$$

$$\left(\frac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2-t_1}} \left(\frac{y_2^t}{y_1^t}\right)^{\frac{1}{t_2-t_1}} =$$

$$\left(\frac{y_1^{t_2}}{y_2^{t_1}}\frac{y_2^t}{y_1^t}\right)^{\frac{1}{t_2-t_1}} =$$

$$\left(\frac{y_2^{t-t_1}}{y_1^{t-t_2}}\right)^{\frac{1}{t_2-t_1}}$$

Thus we have a very useful formula:

$$y(t) = \left(\frac{y_2^{t-t_1}}{y_1^{t-t_2}}\right)^{\frac{1}{t_2-t_1}}$$

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**Example:** A certain function $N(t)$ satisfies the exponential growth law. If $N(3) = 3000$ and $N(6) = 6000$, what is $N(4)$?

**Solution 1(Direct Computation):**

We have $N(t) = N(0)e^{kt}$, and:

$$N(6) = 6000 = N(0)e^{6k}$$

$$N(6) = 3000 = N(0)e^{3k}$$

is easily solved (by dividing the second equation into the first) for $e^{3k} = 2$, and $N(0) = 1500$, so $N(t) = 1500\left(2^{\frac{t}{3}}\right)$, and thus $N(4) = 1500\left(2^{\frac{4}{3}}\right)$

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**Solution 2(Using Formula):** We have $t_1 = 3$, $N_1 = 3000$, $t_2 = 6$, and $N_2 = 6000$, so we have

$$N(t) = \left(\frac{(6000)^{t-3}}{(3000)^{t-6}}\right)^{\frac{1}{6-3}} = \left(\frac{(6000)^{t-3}}{(3000)^{t-3}(3000)^{-3}}\right)^{\frac{1}{3}} = 3000\left(\frac{6000}{3000}\right)^{\frac{t-3}{3}} = 1500\left(2^{\frac{t}{3}}\right)$$

and thus $N(4) = 1500\left(2^{\frac{4}{3}}\right)$

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# Applications:

## (1) Population Growth

**Example:** A bacteria culture starts with 500 bacteria and after 3 hours there are 8000 bacteria.

**(a)** Find an expression for the number of bacteria after $t$ hours.
**(b)** Find the number of bacteria after 4 hours.
**(c)** When will the population reach 30,000?

## Solution 1(Direct Computation):

$y(t) = y(0)e^{kt} = 500e^{kt}$. Since $y(3) = 8000 = 500e^{k(3)}$, we have $e^{3k} = \frac{8000}{500} = 16$, so $3k = \ln 16$, and $k = \frac{\ln 16}{3}$.

**(a)** $y(t) = 500e^{\frac{t \ln 16}{3}} = 500(16)^{\frac{t}{3}}$

**(b)** $y(4) = 500(16)^{\frac{4}{3}}$

**(c)** $y(t) = 30000 = 500(16)^{\frac{t}{3}}$ if $(16)^{\frac{t}{3}} = \dfrac{30000}{500} = 60$,

so we must have $\dfrac{t}{3} \ln 16 = \ln 60$, or $t = 3\dfrac{\ln 60}{\ln 16}$

## Solution 2(Using Formula):

We have $t_1 = 0$, $y_1 = 500 = y(0)$, $t_2 = 3$, and $y_2 = 8000$, so we have

$$y(t) = \left(\frac{(8000)^{t-0}}{(500)^{t-3}}\right)^{\frac{1}{3-0}} = \left(\frac{(8000)^t}{(500)^t(500)^{-3}}\right)^{\frac{1}{3}} = 500\left(\frac{8000}{500}\right)^{\frac{t}{3}} = 500(16)^{\frac{t}{3}}$$

**Example:** The population of Saskatoon is projected to be $P(t) = 200000e^{0.01(t-1998)}$, where $t$ is measured in years.

The population in 2010 will be $P(2010) = 200000e^{0.01(2010-1998)} = 200000e^{0.01(12)} = 200000e^{0.12}$
$\doteq 200,000(1.1274968) = 225,500$,

and in 2020 it will be $P(2020) = 200000e^{0.01(2020-1998)} = 200000e^{0.01(22)} = 200000e^{0.22}$
$\doteq 200,000(1.2460767) = 249,215$

## (2) Radioactive Decay

**Example:** For the radioactive isotope Radium-226 the value of $k$ is $-0.0004359$, assuming that the unit of time is a year. If we had 1000 grams of it (which could be very dangerous) right now, then in $t$ years we would have

$x(t) = 1000e^{-0.0004359t}$ grams left. Thus in one year we would have

$x(1) = 1000e^{-0.0004359(1)} = 1000(0.99956) = 999.56$ grams, so that 0.44 grams would have disappeared.

In 100 years there will be

$x(100) = 1000e^{-0.0004359(100)} = 1000(0.95734) = 957.34$ grams, so that 43 grams would have disappeared.

In 1000 years there will be

$x(1000) = 1000e^{-0.0004359(1000)} = 1000(0.6466554) = 646.66$ grams, so that 353 grams would have disappeared.

In 1590 years there will be $x(1590) = 1000e^{-0.0004359(1590)} = 1000(0.5) = 500$ grams, so that 500 grams would have disappeared.

The number 1590 is called the **half-life** of the isotope.

**Problem:** A radioactive isotope is weighed in a lab. At time $t_1$ there are $y_1$ grams present, and at time $t_2$ there are $y_2$ grams. Find a formula for its half-life.

**Solution (Using Formula):**

$y(t) = \left(\dfrac{y_2^{t-t_1}}{y_1^{t-t_2}}\right)^{\frac{1}{t_2-t_1}}$, so $y(t_1) = y_1$.

We find the value of $t$ for which $y(t) = \dfrac{1}{2}y_1$:

We must have $\dfrac{1}{2}y_1 = \left(\dfrac{y_2^{t-t_1}}{y_1^{t-t_2}}\right)^{\frac{1}{t_2-t_1}}$. We take logarithms:

$-\ln 2 + \ln y_1 = \dfrac{1}{t_2 - t_1}\left[(t - t_1)\ln y_2 - (t - t_2)\ln y_1\right]$, so

$-(t_2 - t_1)\ln 2 + (t_2 - t_1)\ln y_1 = (t - t_1)\ln y_2 - (t - t_2)\ln y_1$, or

$-(t_2-t_1)\ln 2+t_2\ln y_1-t_1\ln y_1 = t(\ln y_2-\ln y_1)-t_1\ln y_2+t_2\ln y_1$, or

$-(t_2 - t_1)\ln 2 + t_1(\ln y_2 - \ln y_1) = t(\ln y_2 - \ln y_1)$, or

$t = \dfrac{-(t_2 - t_1)\ln 2 + t_1(\ln y_2 - \ln y_1)}{\ln y_2 - \ln y_1} = t_1 + \dfrac{(t_2 - t_1)\ln 2}{\ln y_1 - \ln y_2}$.

Thus the half-life is $\boxed{t_{\frac{1}{2}} = \dfrac{t_2 - t_1}{\ln y_1 - \ln y_2} \ln 2}$

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## (3) Continuously Compounded Interest

Exponential functions are often used to model the growth of invested funds by approximating the compounding of interest.

If $P_0$ is invested initially at an annual rate of $100i\%$ compounded $n$ times per year, it will grow to the value

$P_0\left(1 + \dfrac{i}{n}\right)^n$ in one tear, and to the value $P_0\left(1 + \dfrac{i}{n}\right)^n t$ in $t$ years.

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} P_0 \left(1 + \frac{i}{n}\right)^n t =$$

$$P_0 \left(\lim_{n\to\infty} \left(1 + \frac{i}{n}\right)^n\right)^t =$$

$$P_0 \left(\lim_{n\to\infty} e^{\ln\left(1+\frac{i}{n}\right)^n}\right)^t =$$

$$P_0 \left(\lim_{n\to\infty} e^{n\ln\left(1+\frac{i}{n}\right)}\right)^t =$$

$$P_0 \left(\lim_{n\to\infty} e^{\frac{\ln\left(1+\frac{i}{n}\right)}{\frac{1}{n}}}\right)^t = \text{( by L'Hopital)}$$

$$P_0 \left(\lim_{n\to\infty} e^{\frac{\frac{-\frac{i}{n^2}}{1+\frac{i}{n}}}{-\frac{1}{n^2}}}\right)^t =$$

$$P_0\left(\lim_{n\to\infty} e^{\frac{i}{1+\frac{i}{n}}}\right)^t =$$

$$P_0\left(e^{\lim_{n\to\infty}\frac{i}{1+\frac{i}{n}}}\right)^t = P_0e^{it}$$

so we have $\boxed{P(t) = P_0e^{it}}$

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## (4) Mixing of Chemicals

A tank with volume $V$ is full of water in which a chemical is dissolved at a concentration of $c_0$ grams per litre. Keeping the amount of solution in the tank constant, fresh water is added to the tank and mixed thoroughly with the solution which is drained out of the tank at the same rate, $r$ litres per minute. What is the concentration of chemical in the tank as a function of time $t$?

**Solution:** Let $y$ be the amount of chemical in the tank at time $t$, and let $c(t)$ be the concentration of chemical in the tank at time $t$. Then $c(t) = \frac{y}{V}$. The rate of change of $y$ is given by

$y' = -rc(t) = -r\frac{y}{V} = -\frac{r}{V}y$, so

$$\frac{dy}{dt} = -\frac{r}{V}y$$

which becomes, on separation of variables:

$$\frac{dy}{y} = -\frac{r}{V}dt$$

which can be integrated as above:

$$\int \frac{dy}{y} = \int -\frac{r}{V}dt$$

and thus

$$\ln|y| = -\frac{r}{V}t + C$$

Since $y \geq 0$, we can remove the absolute value signs:

$$\ln y = -\frac{r}{V}t + C$$

Taking exponentials, we get

$$y = e^C e^{-\frac{r}{V}t} = y(0)e^{-\frac{r}{V}t}$$

if we let $f = \frac{V}{r}$, the *flush time*, or the length of time it takes for one volume $V$ of the tank to pass through it, we get

$$\boxed{y(t) = y(0)e^{-\frac{t}{f}}}$$

Expressed in terms of the concentration, we have:

$$\boxed{c(t) = \frac{y(0)}{V}e^{-\frac{t}{f}}}$$

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**Example: (Blue Stewart 3.5-21)** A tank contains 1500 litres of brine with a concentration of 0.3 kg per litre. In order to dilute the solution, pure water is run into the tank at the rate of 20 litres per minute and the resulting solution, which is stirred continuously, runs out at the same rate.

(a) How many kilograms of salt remains after 30 minutes?
(b) When will the concentration be reduced to 0.2 kilograms per litre?

**Solution:**

Let $x(t)$ kg be the amount of salt in the tank at time $t$, and let $c(t)$ be the concentration of salt in the tank at time $t$. We have $c(t) = \dfrac{x(t)\text{ kg}}{1500\ell}$, and

$$x'(t) = -c(t)20\frac{\ell}{\text{min}} = \left[-\frac{x(t)\text{ kg}}{1500\ell}\right]20\frac{\ell}{\text{min}} = -\frac{x(t)}{75}\frac{\text{kg}}{\text{min}}$$

$x(t) = ce^{kt}$, so $x'(t) = cke^{kt} = -\frac{x(t)}{75} = -\frac{ce^{kt}}{75} = -\frac{c}{75}e^{kt}$ or $cke^{kt} = -\frac{c}{75}e^{kt}$. Thus $k = -\frac{1}{75}$, and therefore $x(t) = ce^{-\frac{t}{75}}$. Since $x(0) = 450$, we have $c = 450$. Thus $x(t) = 450e^{-\frac{t}{75}}$.

(a) $x(30) = 450e^{-\frac{30}{75}} = 450e^{-\frac{2}{5}}$

(b) $300 = x(t) = 450e^{-\frac{t}{75}}$ if $e^{-\frac{t}{75}} = \frac{2}{3}$ or $-\frac{t}{75} = \ln 2 - \ln 3$. Therefore

$t = 75(\ln 3 - \ln 2)$

**Example: (Blue Stewart 3.5-22)** As before, but instead of pure water, brine with a concentration of 0.1 kg of salt per liter is used.

**Solution:**

Let $x(t)$ kg be the amount of salt in the tank at time $t$, and let $c(t)$ be the concentration of salt in the tank at time $t$. We have $c(t) = \dfrac{x(t)\text{ kg}}{1500\ell}$, and

$$x'(t) = \left[0.1\frac{\text{kg}}{\ell} - c(t)\right]20\frac{\ell}{\text{min}} = \left[0.1\frac{\text{kg}}{\ell} - \frac{x(t)\text{ kg}}{1500\ell}\right]20\frac{\ell}{\text{min}} = \left[2 - \frac{x(t)}{75}\right]$$

$x(t) = 150 + ce^{kt}$, so $x'(t) = cke^{kt} = 2 - \frac{x(t)}{75} = 2 - \frac{150+ce^{kt}}{75} = -\frac{c}{75}e^{kt}$

or $cke^{kt} = -\frac{c}{75}e^{kt}$. Thus $k = -\frac{1}{75}$, and therefore $x(t) = 150 + ce^{-\frac{t}{75}}$. Since $x(0) = 450$, we have $450 = 150 + ce^{-\frac{0}{75}}$ so $c = 300$. Thus $x(t) = 150 + 300e^{-\frac{t}{75}}$.

(a) $x(30) = 150 + 300e^{-\frac{30}{75}} = 150 + 300e^{-\frac{2}{5}}$

(b) $300 = x(t) = 150 + 300e^{-\frac{t}{75}}$ if $e^{-\frac{t}{75}} = \frac{1}{2}$ or $-\frac{t}{75} = -\ln 2$. Therefore $t = 75 \ln 2$

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## (5) Newton's Law of Temperature Change

When a small object with initial temperature $T_0$ is introduced into a temperature controlled environment whose temperature, called the *ambient* temperature, is kept at $A$, **Newton's Law** says that the rate of change of of the temperature $T$ of the small object is proportional to the difference between $T$ and $A$: $\frac{dT}{dt} = k(T - A)$ where $|k|$ is called the **thermal coefficient** of the small object.

Separating variables, we get $\frac{dT}{T - A} = kdt$, $\int \frac{dT}{T - A} = \int kdt$,

$\ln |T - A| = kT + C$, $|T - A| = e^C e^{kt}$

If we assume that $T_0 > A$, then we have $T > A$, and $|T - A| = T - A$, and $k < 0$, so that

$T - A = e^C e^{kt}$ or $T = A + e^C e^{kt}$. Since $T = T_0$ when $t = 0$, we have $T_0 = A + e^C e^{k0} = A + e^C$, so $e^C = T_0 - A$ and thus $T = A + (T_0 - A)e^{kt}$. Since $k < 0$, $\lim_{t\to\infty} e^{kt} = 0$, and therefore $\lim_{t\to\infty} T = A$. Because of this, we often write $T_\infty$ instead of $A$:

$$\boxed{T = T_\infty + (T_0 - T_\infty)e^{kt}}$$

In practice, this equation is assumed to be known to hold and the problem is to use physical observations of temperature to determine the constants that appear in it and then to determine when a certain target temperature will be attained.

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**Example: (Green Stewart 9.4-14)** A thermometer is taken from a room where the temperature is 20° to the outdoors, where the temperature is 5°. After one minute the thermometer reads 12°. Use Newton's Law of Cooling to answer the following questions:

(a) What will the reading on the thermometer be after one more minute?

(b) When will the thermometer read 6°?

**Solution:**

We have $T(t) = T_\infty + (T_0 - T_\infty)e^{kt} = 5 + 15e^{kt}$

Since $T(1) = 12$, we have $12 = 5 + 15e^k$, so $e^k = \frac{7}{15}$. Thus $T(t) = 5 + 15\left(\frac{7}{15}\right)^t$.

(a) $T(1) = 5 + 15\left(\frac{7}{15}\right)^2 = 5 + \frac{49}{15} = 8\frac{4}{15}$

(b) $T(t) = 5 + 15\left(\frac{7}{15}\right)^t = 6$ if $\left(\frac{7}{15}\right)^t = \frac{1}{15}$ or $t\ln\left(\frac{7}{15}\right) = -\ln 15$ or

$$t\ln\left(\frac{7}{15}\right) = -\frac{\ln 15}{\ln\left(\frac{7}{15}\right)} = -\frac{\ln 15}{\ln 7 - \ln 15}$$

## (6) Oil Well Depletion

Oil is pumped continuously from a well at a rate proportional to the amount of oil left in the well. Initially there were 1,000,000 barrels of oil in the well; 6 years later, 500,000 barrels remain.

**(a)** Give a formula for the amount of oil, $B(t)$, in the well (in barrels) as a function of time $t$ (in years).

**Solution:**

$B(t) = 1,000,000e^{kt}$ must satisfy $B(6) = 500,000$,
so $B(6) = 1,000,000e^{6k} = 500,000$,
so $e^{6k} = \frac{1}{2}$.

Thus $B(t) = 1,000,000e^{kt} = 1,000,000e^{6k\frac{t}{6}} = 1,000,000(e^{6k})^{\frac{t}{6}} =$

$$1,000,000\left(\frac{1}{2}\right)^{\frac{t}{6}} = 1,000,000\,(2)^{-\frac{t}{6}}$$

**(b)** At what rate was the amount of oil in the well decreasing when there were 600,000 barrels of oil remaining?

Solution: $B'(t) = \ln\frac{1}{2}B(t) = -(\ln 2)600,000$

**(c)** It will no longer be profitable to pump oil from the well when there are fewer than 50,000 barrels remaining. The plan is to pump oil for 24 years, will the well be profitable for this length of time?

**Solution:**

$B(24) = 1,000,000\left(\frac{1}{2}\right)^{\frac{24}{6}} = 1,000,000\left(\frac{1}{2}\right)^{4} = 1,000,000\frac{1}{16} = 62,500 > 50,000.$

the answer is YES! .

## Logistic Growth Equation

Another important and related differential equation is the **logistic growth equation** used to model biological populations:

$y' = ky(M-y)$ where $k > 0$ is a growth rate factor and $M$ is the **carrying capacity** of the ecosystem.

Separation of variables gives:

$$\int \frac{dy}{y(M-y)} = \int k dt = kt + C, \text{ and we have}$$

$$\int \frac{dy}{y(M-y)} = \int \left( \frac{1}{M}\frac{1}{y} + \frac{1}{M}\frac{1}{M-y} \right) dy = \frac{1}{M}\int \frac{dy}{y} + \frac{1}{M}\int \frac{dy}{M-y} =$$

$$\frac{1}{M}\ln|y|\frac{1}{M}\ln|M-y| = \frac{1}{M}\ln\left|\frac{y}{M-y}\right|$$

Thus we have $\ln\left|\frac{y}{M-y}\right| = M(kt+C)$

Taking exponentials, we get:

$$\left|\frac{y}{M-y}\right| = e^{M(kt+C)} = e^{Mkt}e^{MC}$$

Denoting $y(0)$ by $y_0$, we get:

$\left|\dfrac{y_0}{M-y_0}\right| = e^{Mk(0)}e^{MC} = e^{MC}$, so we have

$$\left|\frac{y}{M-y}\right| = \left|\frac{y_0}{M-y_0}\right| e^{Mkt}$$

Since $y$ represents a population, we must have $y > 0$. It is not necessary to assume that $y < M$ in order for this model to work well, but we shall do so for the purposes of this example.

We then have
$\dfrac{y}{M-y} = \dfrac{y_0}{M-y_0}e^{Mkt}$, which may be solved for $y$:

$$y = \frac{y_0 M}{y_0 + (M-y_0)e^{-kMT}}$$