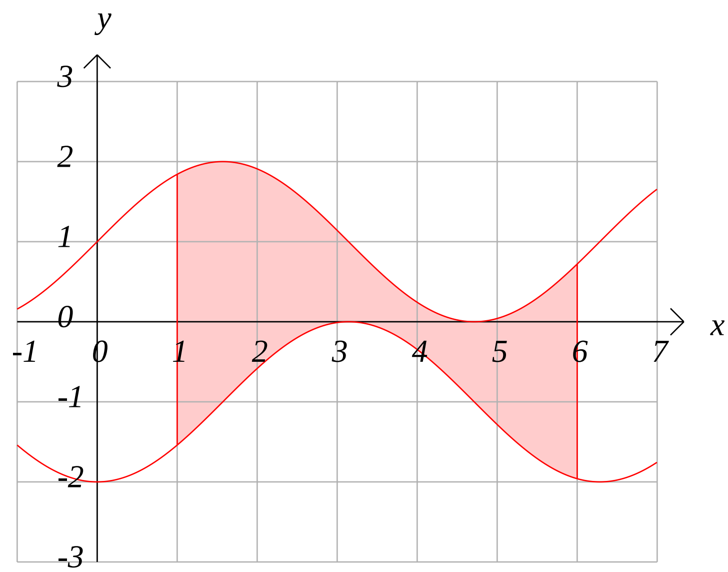
Areas Between Curves

If $f(x) \geq g(x)$ on the interval $[a, b]$, then to find the area A between the graphs of $y = f(x)$ and $y = g(x)$ from a to b we simply evaluate

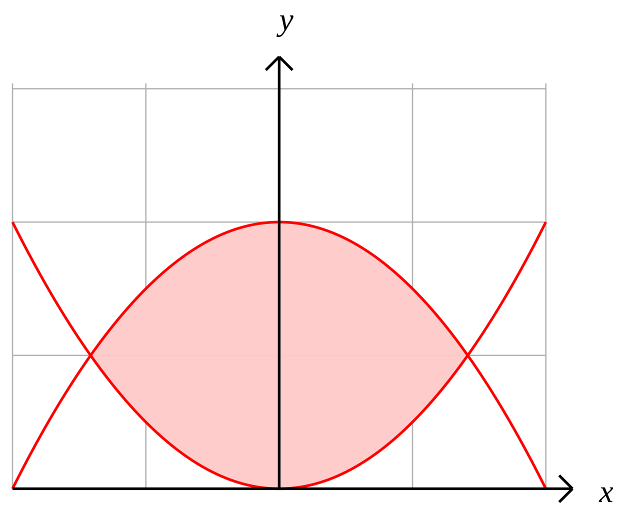
$$A = \int_a^b [f(x) - g(x)]\, dx$$



In practice, difficulties arise from the form or statement of a problem.

For example, the problem "Find the area between the curves $y = x^2$ and $y = 1 - x^2$", if interpreted strictly, would have answer ∞. Yet many people would state such a problem believing that they are asking the question:

" What is the area of the region of the area consisting of points which both lie above the curve $y = x^2$ and below the curve $y = 1 - x^2$?"



To solve this problem, we need to find the points of intersection of the two curves:

$x^2 = 1 - x^2$ if $2x^2 = 1$ or $x^2 = \frac{1}{2}$, so the curves intersect when $x = -\frac{\sqrt{2}}{2}$ and $x = \frac{\sqrt{2}}{2}$, so in our area integral we take $a = -\frac{\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}$:

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left[(1 - x^2) - x^2\right] dx = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 1 - 2x^2 dx =$$

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 1 - 2x^2 dx = x - \frac{2}{3}x^3 \Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} =$$

$$\left[\left(\frac{\sqrt{2}}{2}\right) - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] - \left[\left(-\frac{\sqrt{2}}{2}\right) - \frac{2}{3}\left(-\frac{\sqrt{2}}{2}\right)^3\right] =$$

$$\left[\frac{\sqrt{2}}{2} - \frac{2}{3}\frac{2\sqrt{2}}{8}\right] - \left[-\frac{\sqrt{2}}{2} - \frac{2}{3}\left(-\frac{-2\sqrt{2}}{8}\right)\right] = \frac{\sqrt{2}}{2}\left[1 - \frac{1}{3}\right] + \frac{\sqrt{2}}{2}\left[1 - \frac{1}{3}\right] =$$

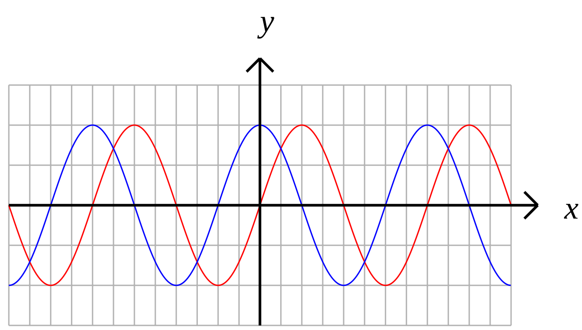
$$\frac{2\sqrt{2}}{3}$$

Note that we can simplify the calculation by making use of the fact that we have symmetry about the y-axis:

$$A = 2\int_0^{\frac{\sqrt{2}}{2}} 1 - 2x^2 dx = 2\left(x - \frac{2}{3}x^3\right)\Bigg|_0^{\frac{\sqrt{2}}{2}} =$$

$$2\left(\frac{\sqrt{2}}{2} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right) = \sqrt{2}\left(1 - \frac{1}{3}\right) = \frac{2\sqrt{2}}{3}$$

Problem: Find the area of the simple regions lying between the intersections of the curves $y = \sin x$ and $y = \cos x$



We have to be very careful to make sure that the function we take for f lies above the function g on the interval $[a, b]$. We let $a = \frac{\pi}{4}$, $b = \frac{5\pi}{4}$, $f(x) = \sin x$, and $g(x) = \cos x$, so that

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\sin x - \cos x] dx = (-\sin x - \cos x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} =$$

$$\left(-\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) - \left(-\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) =$$

$$\left(-\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

Changing Perspective: functions of y

Suppose we have two functions of y like $f(y) = |y|$ and $g(y) = y^2$ which intersect at c and d, (-1 and 1 in this example) and wish to find the area between them.

We use the formula

$$A = \int_c^d [f(y) - g(y)]dy$$

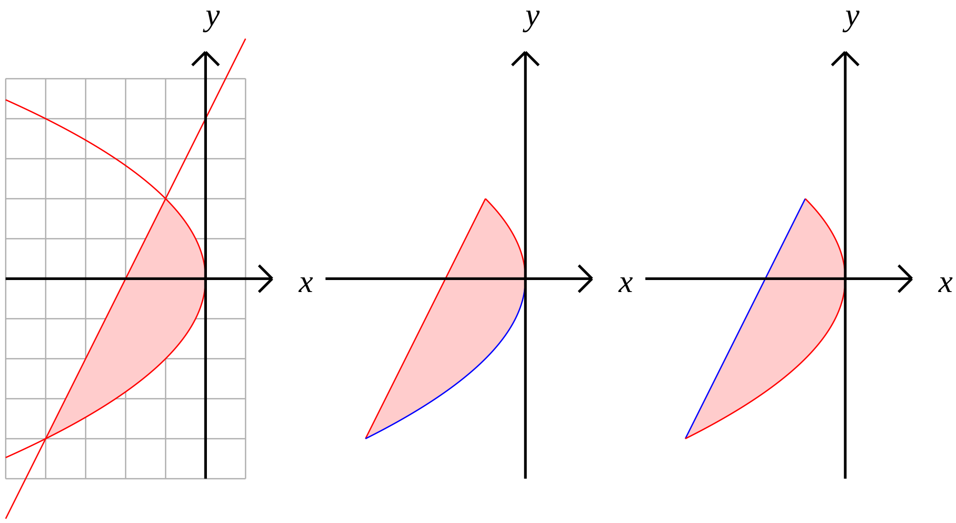
In our example we have

$$A = \int_{-1}^{1} [|y| - y^2]dy = 2\int_0^1 [|y| - y^2]dy = 2\int_0^1 [y - y^2]dy =$$

$$2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

Example: Find the area of the region bounded by the given curves by two methods:
(a) integrating with respect to x, (b) integrating with respect to y, if:

$4x + y^2 = 0,\ y = 2x + 4$



Solution: (a) The upper boundary of the region is the graph of the somewhat complicated function

$$f(x) = \begin{cases} 2x+4 & \text{if } -4 \le x \le -1 \\ \sqrt{-4x} & \text{if } -1 \le x \le 0 \end{cases}$$

while the lower part is the graph of $y = -\sqrt{-4x}, -4 \le x \le 0$.

The area is $A = \int_{-4}^{0} [f(x) - g(x)]\, dx =$

$$\int_{-4}^{-1} [f(x) - g(x)]\, dx + \int_{-1}^{0} [f(x) - g(x)]\, dx =$$

$$\int_{-4}^{-1} \left[2x + 4 - (-\sqrt{-4x})\right] dx + \int_{-1}^{0} \left[\sqrt{-4x} - (-\sqrt{-4x})\right] dx =$$

$$\int_{-4}^{-1} 2x + 4 + 2(-x)^{\frac{1}{2}} dx + 2\int_{-1}^{0} 2(-x)^{\frac{1}{2}} dx =$$

$$x^2 + 4x\Big|_{-4}^{-1} + 2\int_{-4}^{-1} (-x)^{\frac{1}{2}} dx + 4\int_{-1}^{0} (-x)^{\frac{1}{2}} dx =$$

Sidetrack: We need to find $\int (-x)^{\frac{1}{2}} dx$ by making the substitution $u = -x$, $dx = -du$:

$$\int (-x)^{\frac{1}{2}} dx = \int u^{\frac{1}{2}} (-du) = -\int u^{\frac{1}{2}} du = -\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{2}{3}(-x)^{\frac{3}{2}} + C$$

Thus we get

$$A = x^2 + 4x\Big|_{-4}^{-1} + 2\int_{-4}^{-1}(-x)^{\frac{1}{2}}dx + 4\int_{-1}^{0}(-x)^{\frac{1}{2}}dx =$$

$$x^2 + 4x\Big|_{-4}^{-1} + 2\left(-\frac{2}{3}(-x)^{\frac{3}{2}}\right)\Big|_{-4}^{-1} + 4\left(-\frac{2}{3}(-x)^{\frac{3}{2}}\right)\Big|_{-1}^{0} =$$

$$\Big((-1)^2 + 4(-1)\Big) - \Big((-4)^2 + 4(-4)\Big) + \left[-\frac{4}{3}(-(-1))^{\frac{3}{2}} - \left(-\frac{4}{3}(-(-4))^{\frac{3}{2}}\right)\right] + \left[4\frac{2}{3}(-0)^{\frac{3}{2}} - 4\frac{-2}{3}(-(-1))^{\frac{3}{2}}\right] =$$

$$(1-4) - (16-16) + \left[-\frac{4}{3} + \frac{4}{3}(4)^{\frac{3}{2}}\right] + \left[0 - \frac{-8}{3}\right] =$$

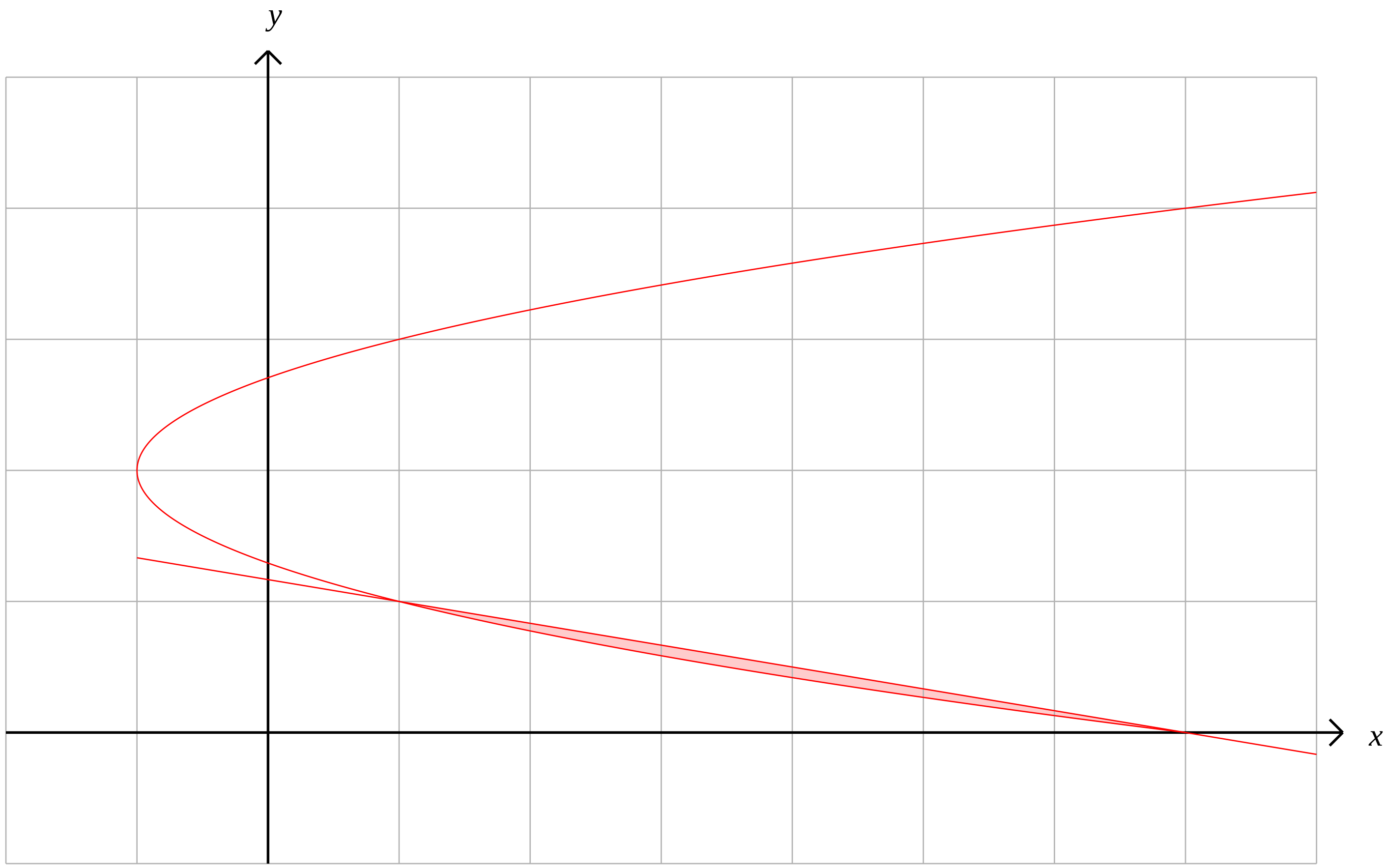
$$-3 + \left[-\frac{4}{3} + \frac{4}{3}8\right] + \frac{8}{3} = 9$$

(b) We first solve the two equations $4x + y^2 = 0$, and $y = 2x + 4$ for x as a function of y and get

$$x = -\frac{y^2}{4} \quad \text{and} \quad x = \frac{y-4}{2}$$

Thus we have $A = \int_{-4}^{2} \left[-\frac{y^2}{4} - \frac{y-4}{2} \right] dy = \int_{-4}^{2} -\frac{y^2}{4} - \frac{y}{2} + 2dy =$
$-\frac{y^3}{12} - \frac{y^2}{4} + 2y \bigg|_{-4}^{2} = \left(-\frac{2^3}{12} - \frac{2^2}{4} + 2(2) \right) - \left(-\frac{(-4)^3}{12} - \frac{(-4)^2}{4} + 2(-4) \right)$
$\left(-\frac{8}{12} - \frac{4}{4} + 4 \right) - \left(\frac{-64}{12} - \frac{16}{4} - 8 \right) = \left(-\frac{2}{3} - 1 + 4 \right) - \left(-\frac{-16}{3} - 4 - 8 \right) =$
$-\frac{2}{3} + 3 - \frac{16}{3} + 12 = 15 - \frac{18}{3} = 9$

Example: Find the area of the region bounded by the given curves by two methods:
(a) integrating with respect to x, (b) integrating with respect to y, if:
$x + 1 = 2(y - 2)^2$, $x + 6y = 7$



Solution: (a) The two curves intersect at the points $(1, 1)$ and $(7, 0)$, so we have

$$A = \int_1^7 \left[\frac{7-x}{6} - \left(2 - \sqrt{\frac{x+1}{2}}\right)\right] dx = \int_1^7 -\frac{5}{6} - \frac{x}{6} + \sqrt{\frac{x+1}{2}} dx =$$

$$-\frac{5}{6}x - \frac{x^2}{12} + \frac{2}{3\sqrt{2}}(x+1)^{\frac{3}{2}}\Big|_1^7 =$$

$$\left(-\frac{5}{6}7 - \frac{7^2}{12} + \frac{2}{3\sqrt{2}}(7+1)^{\frac{3}{2}}\right) - \left(-\frac{5}{6}1 - \frac{1^2}{12} + \frac{2}{3\sqrt{2}}(1+1)^{\frac{3}{2}}\right) =$$

$$\left(-\frac{35}{6} - \frac{49}{12} + \frac{2}{3\sqrt{2}}(8)^{\frac{3}{2}}\right) - \left(-\frac{5}{6} - \frac{1}{12} + \frac{2}{3\sqrt{2}}(2)^{\frac{3}{2}}\right) =$$

$$\left(-\frac{119}{12} + \frac{2}{3\sqrt{2}}8\sqrt{8}\right) - \left(-\frac{11}{12} + \frac{2}{3\sqrt{2}}2\sqrt{2}\right) =$$

$$\left(-\frac{108}{12} + \frac{2}{3\sqrt{2}}8\left(2\sqrt{2}\right)\right) - \left(\frac{4}{3}\right) = -9 + \frac{32}{3} - \frac{4}{3} = \frac{1}{3}$$

(b) $A = \int_0^1 \left[(7-6y) - \left(2(y-2)^2 - 1\right)\right] dy =$

$$\int_0^1 7 - 6y - \left(2(y^2 - 4y + 4) - 1\right) dy =$$

$$\int_0^1 7 - 6y - \left(2y^2 - 8y + 8 - 1\right) dy =$$

$$\int_0^1 -2y^2 + 2y dy = -2\frac{y^3}{3} + 2\frac{y^2}{2}\Bigg|_0^1 = -\frac{2}{3} + 1 = \frac{1}{3}$$

Two strategies become clear from looking at these two examples:

First: if possible, avoid functions whose definitions must involve different formulas on different intervals.

Second: choose the integral that will have the simplest expression.

In both of the examples just looked at, it was best to integrate with respect to y. It is easy to find examples where it is better to integrate with respect to x: just rotate the above examples by 90 degrees!