Area:An Introduction to Integration

According to the Oxford English Dictionary, the word **calculus** has two meanings:

(1) a stone, or concretion in some part of the body,

(2) a particular method of calculation.

Thus **Differential Calculus** is the method used to analyse how rapidly a function is changing. The basic idea is to look at the ratio

$$\frac{f(x) - f(a)}{x - a}$$

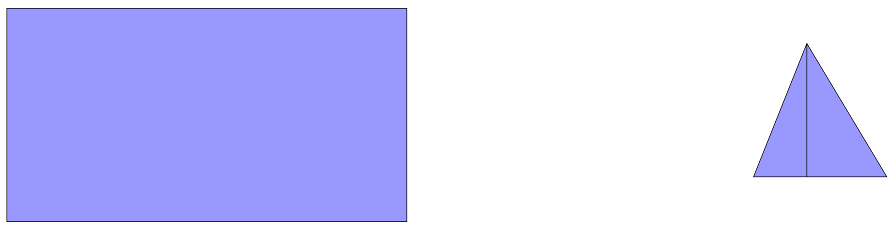
as x approaches a. In so doing, we are focussing on the point $(a, f(a))$, and "differentiating" between it and nearby points $(x, f(x))$. The geometric interpretation of this ratio as the slope of a line is quite useful in understanding the concept, even when the function f models something quite non-geometrical, such as the rate of change of temperature, or the flow of electrons past an electrical junction.

On the other hand, **Integral Calculus** is the method used to predict how much a function will change over an interval in its domain if we know its rate of change, or derivative. Thus we may have already seen how to predict how money compounded continuously will grow in a bank account, how to predict how the temperature of an object being heated or cooled will change, and how to predict how long it will take to drain tanks of various shapes and sizes. The purely mathematical problem may be stated as that of finding an "antiderivative" — a continuous function F whose "derivative" is some given function f — which is not necessarily continuous.

The key mathematical idea is that the values of the derivative over the whole of an interval, and not just near a point, are important. We take the information at a large number of points(okay, for the mathematical purist, the number is $\mathfrak{c}$, the cardinality of the continuum) and "**integrate**" it to get the amount of change.

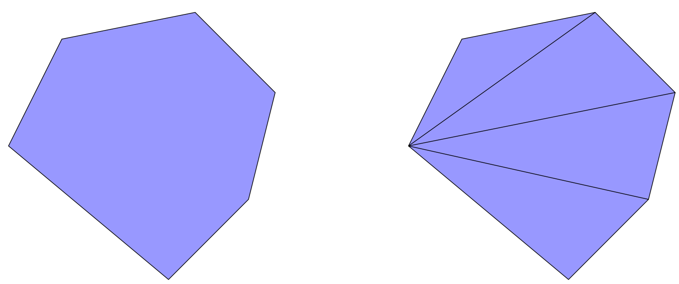
As with Differentiation, the geometric interpretation is useful. We think of integration in terms of area underneath a positive curve over an interval. When dealing with negative curves, the concept of signed area will naturally arise. The problem that then arises is that the area under a curve, while quite intuitive, is not easily defined. Let us review what we really know from basic geometry:

(1) The area of a rectangle is equal to the product of its base times its height.



(2) The area of a triangle is one half its base times its height.

(3) The area of any region bounded by straight lines may be computed by cutting the region up into rectangles and triangles and summing the resulting areas.

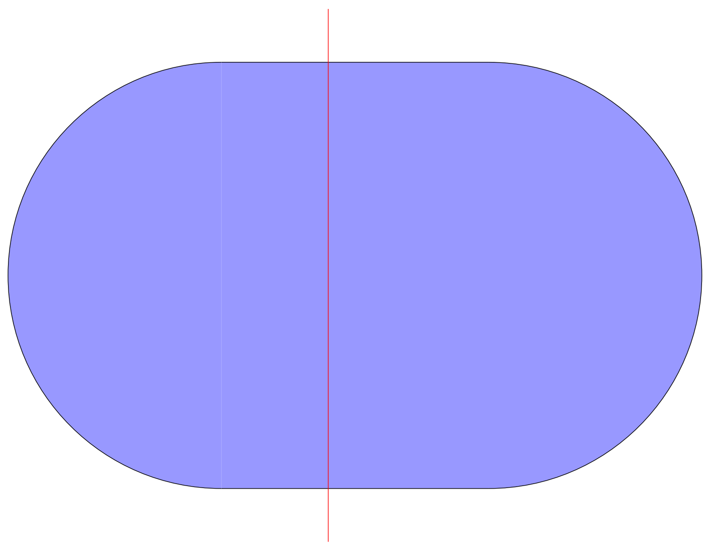


Let us agree on some of the properties that the "area" A of a region $\mathcal{R}$ should have:

(1) A should be greater than or equal to the area of any region that $\mathcal{R}$ contains.

(2) A should be less than or equal to the area of any region that contains $\mathcal{R}$.

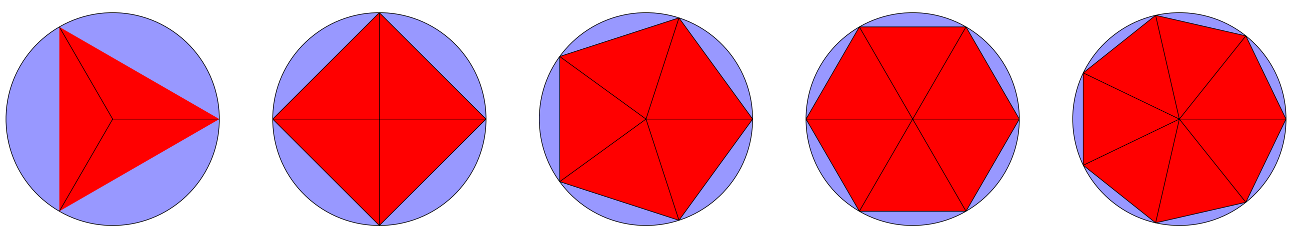
(3) If $\mathcal{R}$ is divided by a straight line into two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ with respective areas A_1 and A_2 then $A = A_1 + A_2$.



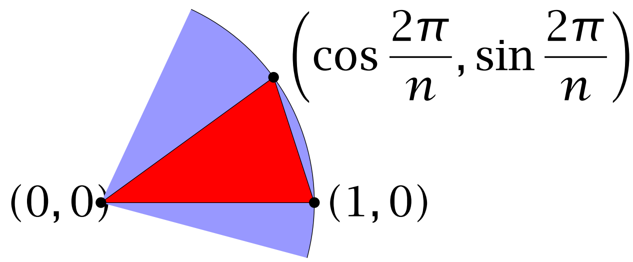
Area of the Unit Circle

Students may recall the formula ($A = \pi r^2$) for the area of a circle of radius r, but would be hard put to prove that a circle can be dissected into pieces that can be arranged to form a square. (Mainly because it's impossible, not because of any failing of the student!) In order to even define the area of regions with curved boundaries we are going to have to develop a lot of mathematical machinery.

Example: Inscribed Polygons Let us try to estimate the area of the unit circle (i.e., of radius 1) by finding the areas of the largest (a)triangle (b) square (c) pentagon (d) hexagon (e) septagon (f) octagon (g) n-gon that may be inscribed inside the unit circle.

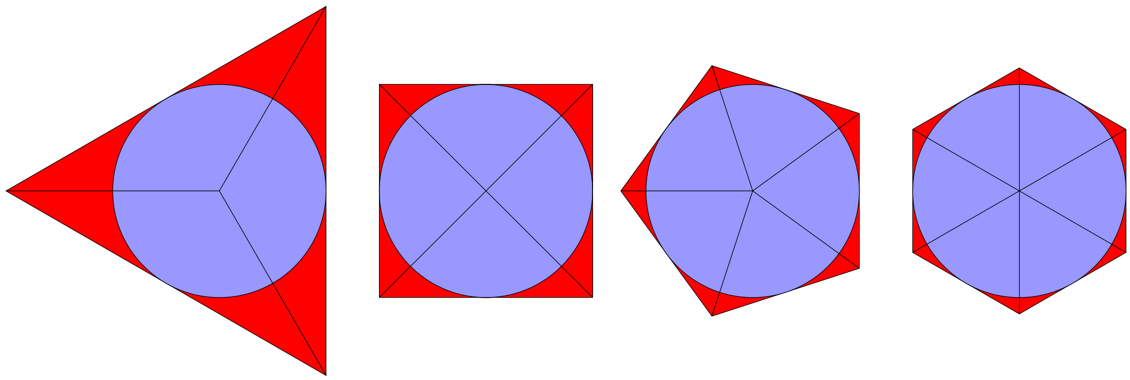


Let a_n be the area of such an n-gon, and let t_n be the triangle with vertices $(0,0)$, $(1,0)$ and $\left(\cos\dfrac{2\pi}{n}, \sin\dfrac{2\pi}{n}\right)$, so that a_n is n times the area of t_n, which is $\frac{1}{2}\sin\frac{2\pi}{n}$, and thus $a_n = \frac{n}{2}\sin\frac{2\pi}{n} = \frac{\sin\frac{2\pi}{n}}{\frac{2}{n}} = \pi\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}$.



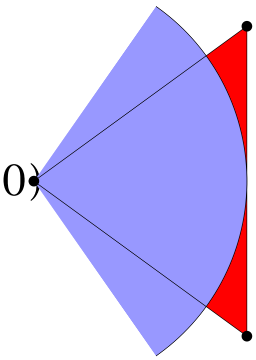
We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \pi\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi$

Example: Exscribed Polygons Let us try to estimate the area of the unit circle (i.e., of radius 1) by finding the areas of the smallest (a)triangle (b) square (c) pentagon (d) hexagon (e) septagon (f) octagon (g) n-gon that may be exscribed outside the unit circle.



Let A_n is the area of such an n-gon, and let T_n be the triangle with vertices $(0,0)$, $\left(1, \tan\frac{\pi}{n}\right)$, and $\left(1, -\tan\frac{\pi}{n}\right)$, so that A_n is n times the area of T_n, which is $\tan\frac{\pi}{n}$, and thus

$$A_n = n\tan\frac{\pi}{n} = \frac{\tan\frac{\pi}{n}}{\frac{1}{n}} = \frac{\pi}{\cos\frac{\pi}{n}}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}.$$



$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{\pi}{\cos\frac{\pi}{n}} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = \pi$$

We will always have $a_n < \pi < A_n$. Tables of values are interesting:

n	a_n	A_n
3	1.299	5.200
4	2	4
5	2.378	3.633
6	2.598	3.464
7	2.736	3.371
8	2.828	3.314

k	$n = 2^k$	a_n	A_n
2	4	2	4
3	8	2.828	3.314
4	16	3.061467	3.182597
5	32	3.121445	3.151724
6	64	3.136548	3.144118
7	128	3.140331	3.142223
8	256	3.141277	3.141750
9	512	3.141513	3.141632
10	1024	3.141572	3.141602

Note that $\pi \doteq 3.141592653589793$.

How π was calculated before calculus.

The first purely mathematical (as opposed to empirical measurement) attempt to calculate π was undertaken by Archimedes of Syracuse about 200 B.C. who used a value of $n = 96$. He did not have an efficient system of numerical notation. Before the development of decimal notation in the Middle Ages, and logarithms and Calculus in the Seventeenth Century, one of the few numerical techniques available was that of computing square roots using methods that are now long (and probably best), forgotten.

Substituting $n = 2^m$ into the formulas $a_n = \dfrac{n}{2}\sin\dfrac{2\pi}{n}$ and $A_n = n\tan\dfrac{\pi}{n} = \dfrac{\tan\frac{\pi}{n}}{\frac{1}{n}}$ gives :

$a_{2^m} = \dfrac{2^m}{2}\sin\dfrac{2\pi}{2^m} = 2^{m-1}\sin\dfrac{\pi}{2^{m-1}}$ and $A_{2^m} = 2^m\tan\dfrac{\pi}{2^m}$, so

$$2^{m-1}\sin\frac{\pi}{2^{m-1}} < \pi < 2^m\tan\frac{\pi}{2^m}$$

We can use the half-Angle Formulas

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}, \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}, \text{ and } \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

to compute successive values of a_{2k} and A_{2k}.

Archimedes' Estimate

Letting $\alpha = \frac{\pi}{6}$, and using $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$$\sin\frac{\pi}{12} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}, \ \cos\frac{\pi}{12} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}, \ \tan\frac{\pi}{8} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}},$$

so $a_{12} = a_{2^2 3} = 12\sin\frac{\pi}{12} = 12\frac{\sqrt{2-\sqrt{3}}}{2} = 6\sqrt{2-\sqrt{3}}$,

$A_{12} = 12\tan\frac{\pi}{8} = 12\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$

Next, letting $\alpha = \frac{\pi}{24}$,

$$\sin\frac{\pi}{24} = \sqrt{\frac{1 - \frac{\sqrt{2+\sqrt{3}}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{2 + \sqrt{3}}}}{2}, \cos\frac{\pi}{24} = \sqrt{\frac{1 + \frac{\sqrt{2+\sqrt{3}}}{2}}{2}} = \frac{\sqrt{2 + \sqrt{2 + \cdots}}}{2}$$

$$\tan\frac{\pi}{24} = \sqrt{\frac{2 - \sqrt{2 + \sqrt{4}}}{2 + \sqrt{2 + \sqrt{3}}}},$$

$$\text{so } a_{24} = 12\sin\frac{\pi}{24} = 6\frac{\sqrt{2 - \sqrt{2 + \sqrt{3}}}}{2} < \pi < A_{24} = 24\tan\frac{\pi}{24} = 24\sqrt{\frac{2 - \sqrt{2 + \cdots}}{2 + \sqrt{2 + \cdots}}}$$

A clear but laborious pattern emerges:

$$12\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} < \pi < 48\sqrt{\frac{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}}{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}$$

$$24\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} < \pi < 96\sqrt{\frac{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}},$$

which is where Archimedes stopped 2200 years ago.

Viete's Estimate

Letting $\alpha = \dfrac{\pi}{4}$,

$\sin\dfrac{\pi}{8} = \sqrt{\dfrac{1-\frac{\sqrt{2}}{2}}{2}} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$, $\cos\dfrac{\pi}{8} = \sqrt{\dfrac{1+\frac{\sqrt{2}}{2}}{2}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$, $\tan\dfrac{\pi}{8} = \sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}}$,

so $a_{16} = a_{2^4} = 2^3 \sin\dfrac{\pi}{2^3} = 8\sin\dfrac{\pi}{8} = 8\dfrac{\sqrt{2-\sqrt{2}}}{2} = 4\sqrt{2-\sqrt{2}}$,

$A_8 = A_{2^3} = 2^3 \tan\dfrac{\pi}{8} = 8\sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}} = 8\sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\left(\dfrac{2-\sqrt{2}}{2-\sqrt{2}}\right)} = 8\sqrt{\dfrac{(2-\sqrt{2})^2}{2}} =$
$8\dfrac{2-\sqrt{2}}{\sqrt{2}} = 8(\sqrt{2}-1)$

Next, letting $\alpha = \frac{\pi}{8}$,

$$\sin\frac{\pi}{16} = \sqrt{\frac{1 - \frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}, \cos\frac{\pi}{16} = \sqrt{\frac{1 + \frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2+}}}{2}$$

$$\tan\frac{\pi}{16} = \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},$$

$$\text{so } a_{32} = a_{2^5} = 2^4 \sin\frac{\pi}{2^4} = 16\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2} = 8\sqrt{2-\sqrt{2+\sqrt{2}}},$$

$$A_{16} = A_{2^4} = 2^4 \tan\frac{\pi}{16} = 16\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}} = 16\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\left(\frac{2-\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}\right.}$$

$$16\frac{2-\sqrt{2+\sqrt{2}}}{\sqrt{4-(2+\sqrt{2})}} = 16\frac{2-\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\left(\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right) = 16\frac{2\sqrt{2+\sqrt{2}}-(2+\sqrt{2})}{\sqrt{2}}$$

A clear but laborious pattern emerges.

Viete's Infinite Product

Substituting $n = 2m$ into the formulas $a_n = \frac{n}{2}\sin\frac{2\pi}{n}$ and $A_n = n\tan\frac{\pi}{n} = \frac{\tan\frac{\pi}{n}}{\frac{1}{n}}$ gives :

$a_{2m} = \frac{2m}{2}\sin\frac{2\pi}{2m} = m\sin\frac{\pi}{m}$ and $A_{2m} = 2m\tan\frac{\pi}{2m}$, so

$$m\sin\frac{\pi}{m} < \pi < 2m\tan\frac{\pi}{2m}$$

Let us examine the ratio $\frac{a_{2m}}{a_m} = \frac{m\sin\frac{\pi}{m}}{\frac{m}{2}\sin\frac{2\pi}{m}} = 2\frac{\sin\frac{\pi}{m}}{\sin 2\frac{\pi}{m}} = 2\frac{\sin\frac{\pi}{m}}{2\sin\frac{\pi}{m}\cos\frac{\pi}{m}} = \frac{1}{\cos\frac{\pi}{m}}$

so $a_{2m} = \frac{a_m}{\cos\frac{\pi}{m}}$. Since $a_4 = \frac{4}{2}\sin\frac{2\pi}{4} = 2\sin\frac{\pi}{2} = 2$, we have

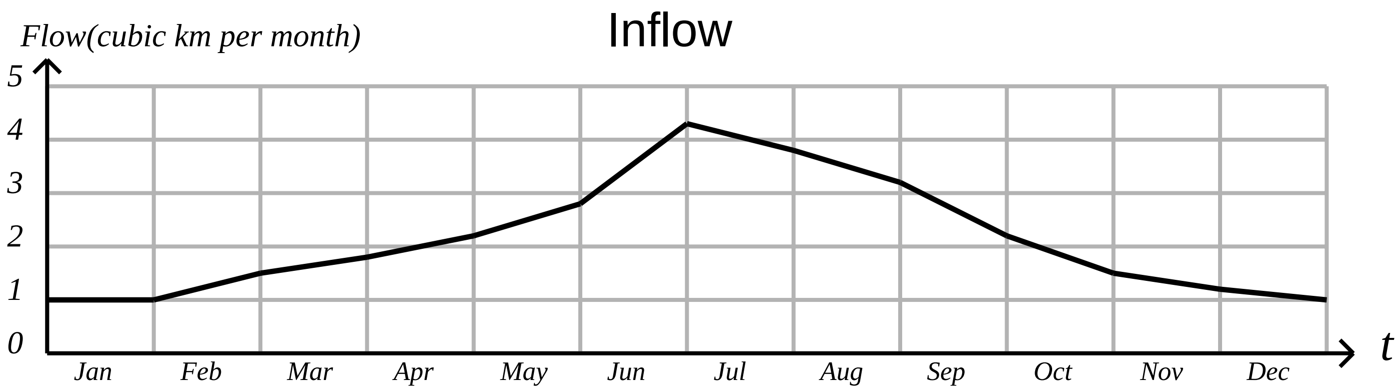
$a_8 = \frac{2}{\cos\frac{\pi}{8}}$, $a_{16} = \frac{2}{\cos\frac{\pi}{8}\cos\frac{\pi}{16}}$, $a_{32} = \frac{2}{\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32}}$,

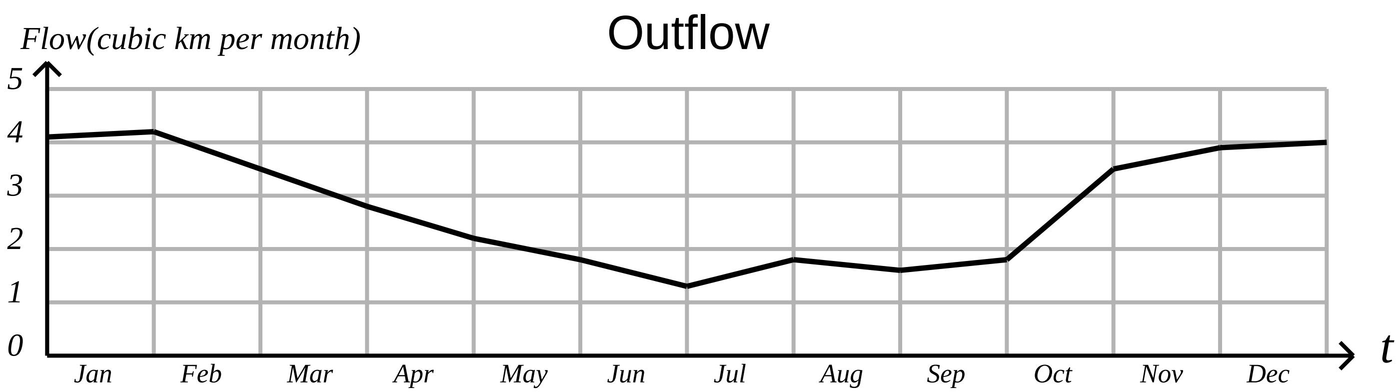
$\pi = \lim_{n\to\infty} a_n = \frac{2}{\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32}\cdots}$

Exercises

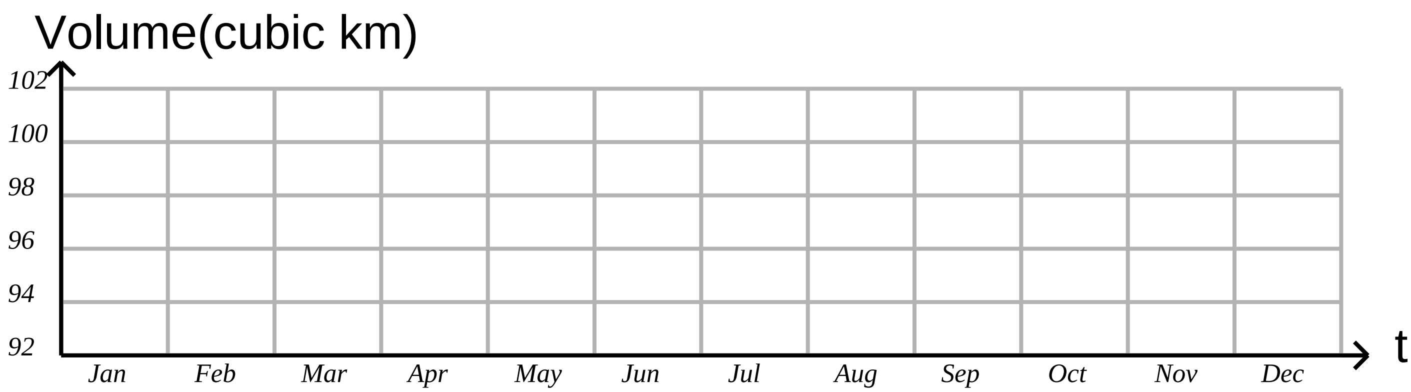
(1) Find the area of the region bounded by the polygon $P_1P_2P_3P_4P_5$ whose vertices are the points $P_1 = (1,0)$, $P_2 = (3,4)$, $P_3 = (5,1)$, $P_4 = (6,-1)$, $P_5 = (2,-3)$.

(2) The graph shown below shows the flow of water into and out of Lake Diefenbaker at the Gardiner Dam for the year 1994.





Sketch a graph of the quantity of water in Lake Diefenbaker, given that on New Year's Day, 1994, there was 200 cubic kilometres of water in Lake Diefenbaker.



Quantity of Water in Lake Diefenbaker