Arc Length

The length of the curve $(x(t), y(t))$ as t varies from t_0 to t_1 is given by

$$\mathcal{L} = \int_{t=t_0}^{t=t_1} \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

If we wish to find the length of a curve which is the graph of a function $y = f(x)$, as x runs from a to b,

we let $x(t) = t$, $y(t) = f(x(t)) = f(x)$ and we get $x'(t) = 1$, and $y'(t) = f'(x(t))x'(t) = f'(x)$, so we have a simple formula for the length:

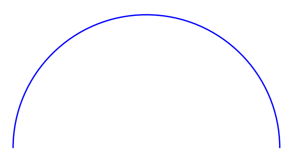
$$\mathcal{L} = \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_a^b \sqrt{1 + (y')^2} dx$$

Similarly, if we have a curve $x = g(y)$ with y running from c to d we get

$$\mathcal{L} = \int_{y=c}^{y=d} \sqrt{1 + (g'(y))^2} dy = \int_c^d \sqrt{1 + (g'(y))^2} dy = \int_c^d \sqrt{1 + (x')^2} dy$$

Example: Consider the curve given by

$x(t) = \cos t,\ y(t) = \sin t\ , 0 \le t \le \pi.$

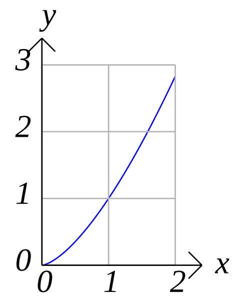


Its length is

$$L = \int_{t=0}^{t=\pi} \sqrt{x'(t)^2 + y'(t)^2} dt = \int_{t=0}^{t=\pi} \sqrt{(-\sin t)^2 + (\cos t)^2} dt =$$

$$\int_{t=0}^{t=\pi} 1 dt = t|_0^{\pi} = \pi$$

Example: Find the length of the curve $y = x^{\frac{3}{2}}, 0 \le a \le x \le b$



We have $\mathcal{L} = \int_a^b \sqrt{1 + (y')^2}dx =$

$$\int_a^b \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}dx =$$

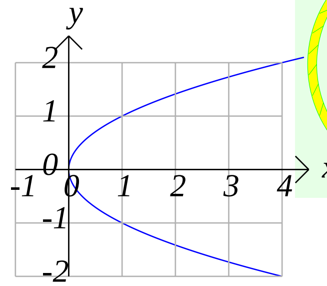
$$\int_a^b \sqrt{1 + \frac{9}{4}x}dx =$$

$$\frac{4}{9} \left. \frac{(1 + \frac{9}{4}x)^{\frac{3}{2}}}{\frac{3}{2}} \right|_a^b =$$

$$\frac{8}{27} \left. \left(1 + \frac{9}{4}x\right)^{\frac{3}{2}} \right|_a^b = \frac{8}{27} \left. \left(\frac{4 + 9x}{4}\right)^{\frac{3}{2}} \right|_a^b = \frac{1}{27} \left. (4 + 9x)^{\frac{3}{2}} \right|_a^b =$$

$$\frac{1}{27}\left[(4 + 9b)^{\frac{3}{2}} - (4 + 9a)^{\frac{3}{2}}\right]$$

Example: Find the length of the curve $x = y^2,\ 0 \le c \le y \le b$



We have $\mathcal{L} = \int_{y=c}^{y=d} \sqrt{1 + (x')^2}\,dy = \int_{y=c}^{y=d} \sqrt{1 + (2y)^2}\,dy = \int_{y=c}^{y=d} \sqrt{1 + 4y^2}\,dy$

Making the substitution $y = \frac{1}{2}\tan\theta$, we have
$dy = \frac{1}{2}\sec^2\theta\,d\theta$, $1 + 4y^2 = 1 + \tan^2\theta = \sec^2\theta$,
$\theta_c = \arctan 2c$ when $y = c$ and $\theta_d = \arctan 2d$ when $y = d$, so

$$\mathcal{L} = \int_{\theta=\theta_c}^{\theta=\theta_d} \sqrt{\sec^2\theta}\,\frac{1}{2}\sec^2\theta\,d\theta = \frac{1}{2}\int_{\theta=\theta_c}^{\theta=\theta_d} \sec^3\theta\,d\theta =$$

$$\frac{1}{2}\left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right]\Bigg|_{\theta=\theta_c}^{\theta=\theta_d} =$$

$$\frac{1}{4}\left[\sec\theta_d\tan\theta_d + \ln|\sec\theta_d + \tan\theta_d|\right] - \frac{1}{4}\left[\sec\theta_c\tan\theta_c + \ln|\sec\theta_c + \tan\theta_c|\right] =$$

$$\frac{1}{4}\left[\sqrt{1+4d^2}\,2d + \ln|\sqrt{1+4d^2} + 2d|\right] - \frac{1}{4}\left[\sqrt{1+4c^2}\,2c + \ln|\sqrt{1+4c^2} + 2c|\right] =$$

$$\frac{d\sqrt{1+4d^2} - c\sqrt{1+4c^2}}{2} + \frac{1}{4}\ln\frac{\sqrt{1+4d^2} + 2d}{\sqrt{1+4c^2} + 2c}$$

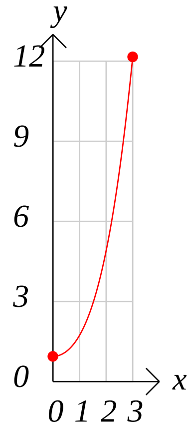
Note that if we let $c = 0$, we get the formula for the distance along the parabola to the point (d^2, d):

$$\mathcal{L} = \frac{d\sqrt{1+4d^2}}{2} + \frac{1}{4}\ln(\sqrt{1+4d^2} + 2d)$$

and if we let $b = d^2$, we get the (equivalent) formula for the distance from (0,0) to $(b, \sqrt{b})$:

$$\mathcal{L} = \frac{\sqrt{b}\sqrt{1+4b}}{2} + \frac{1}{4}\ln\left(\sqrt{1+4b} + 2\sqrt{b}\right)$$

Example: Find the length of the curve $y = \frac{1}{3}\left(x^2+2\right)^{\frac{3}{2}}$, $0 \le x \le 3$



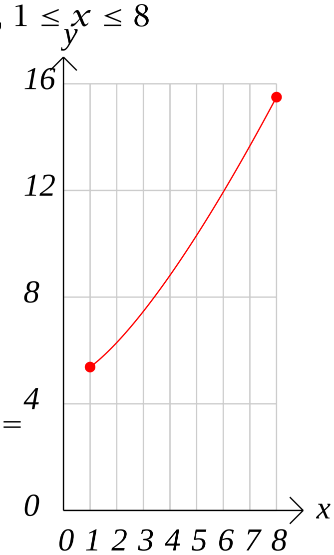
Solution: $y' = \frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{\frac{1}{2}} 2x = x\left(x^2+2\right)^{\frac{1}{2}}$, so

$1+(y')^2 = 1 + x^2(x^2+2) = \left(x^2\right)^2 + 2x^2 + 1 = \left(x^2+1\right)^2$, and

$\mathcal{L} = \int_0^3 \sqrt{(x^2+1)^2}dx = \int_0^3 x^2+1dx = \frac{x^3}{3} + x\Big|_0^3 =$

$\left(\frac{3^3}{3}+3\right) - \left(\frac{0^3}{3}+0\right) = 12$

Example: Find the length of the curve $y = \frac{3}{4}x^{\frac{4}{3}} - \frac{3}{8}x^{\frac{2}{3}} + 5$, $1 \le x \le 8$



Solution:

$y' = \frac{3}{4}\frac{4}{3}x^{\frac{1}{3}} - \frac{3}{8}\frac{2}{3}x^{-\frac{1}{3}} = x^{\frac{1}{3}} - \frac{1}{4}x^{-\frac{1}{3}}$, so

$1+(y')^2 = 1 + \left(x^{\frac{1}{3}} - \frac{1}{4}x^{-\frac{1}{3}}\right)^2 = \left(x^{\frac{1}{3}} + \frac{1}{4}x^{-\frac{1}{3}}\right)^2$, and

$\mathcal{L} = \int_1^8 \sqrt{\left(x^{\frac{1}{3}} + \frac{1}{4}x^{-\frac{1}{3}}\right)^2} dx = \int_1^8 \left(x^{\frac{1}{3}} + \frac{1}{4}x^{-\frac{1}{3}}\right) dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{1}{4}\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\Bigg|_1^8 =$

$\mathcal{L} = \frac{3}{4}x^{\frac{4}{3}} + \frac{3}{8}x^{\frac{2}{3}}\Big|_1^8 = \left(\frac{3}{4}8^{\frac{4}{3}} + \frac{3}{8}8^{\frac{2}{3}}\right) - \left(\frac{3}{4}1^{\frac{4}{3}} + \frac{3}{8}1^{\frac{2}{3}}\right) =$

$\left(\frac{3}{4}2^4 + \frac{3}{8}2^2\right) - \left(\frac{3}{4} + \frac{3}{8}\right) = 12 + \frac{3}{2} - \frac{9}{8} = 12\frac{3}{8}$
