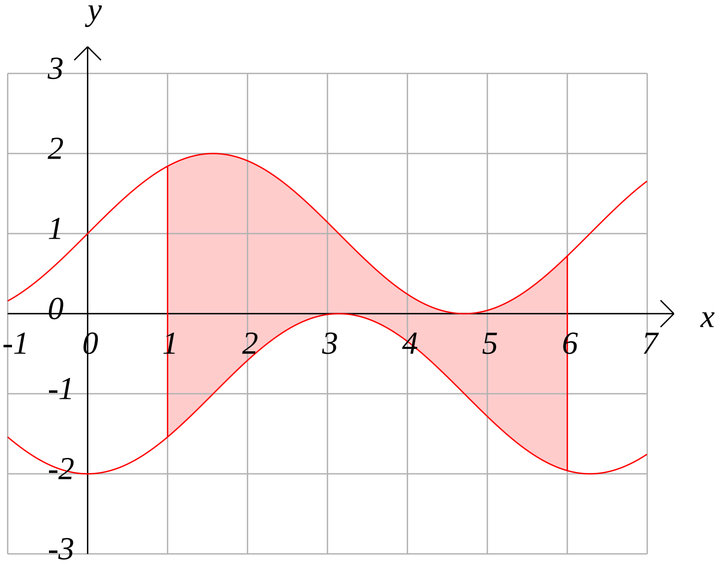
Applications of Integration I

Areas Between Curves

If $f(x) \geq g(x)$ on the interval $[a, b]$, then to find the area A between the graphs of $y = f(x)$ and $y = g(x)$ from a to b we simply evaluate

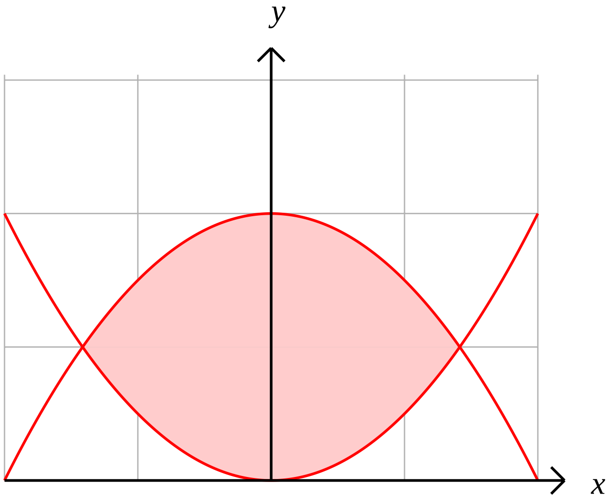
$$A = \int_a^b [f(x) - g(x)]\,dx$$



In practice, difficulties arise from the form or statement of a problem.

For example, the problem "Find the area between the curves $y = x^2$ and $y = 1 - x^2$", if interpreted strictly, would have answer ∞. Yet many people would state such a problem believing that they are asking the question:

" What is the area of the region of the area consisting of points which both lie above the curve $y = x^2$ and below the curve $y = 1 - x^2$?"



To solve this problem, we need to find the points of intersection of the two curves:

$x^2 = 1 - x^2$ if $2x^2 = 1$ or $x^2 = \frac{1}{2}$, so the curves intersect when $x = -\frac{\sqrt{2}}{2}$ and $x = \frac{\sqrt{2}}{2}$, so in our area integral we take

$a = -\frac{\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}$:

$$A = \int_a^b \left[f(x) - g(x)\right] dx = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left[(1 - x^2) - x^2\right] dx = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 1 - 2x^2 dx =$$

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 1 - 2x^2 dx = x - \frac{2}{3}x^3 \Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} =$$

$$\left[\left(\frac{\sqrt{2}}{2}\right) - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] - \left[\left(-\frac{\sqrt{2}}{2}\right) - \frac{2}{3}\left(-\frac{\sqrt{2}}{2}\right)^3\right] =$$

$$\left[\frac{\sqrt{2}}{2} - \frac{2}{3}\frac{2\sqrt{2}}{8}\right] - \left[-\frac{\sqrt{2}}{2} - \frac{2}{3}\left(-\frac{-2\sqrt{2}}{8}\right)\right] = \frac{\sqrt{2}}{2}\left[1 - \frac{1}{3}\right] + \frac{\sqrt{2}}{2}\left[1 - \frac{1}{3}\right] =$$

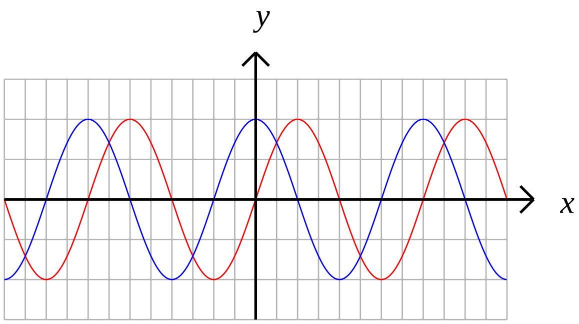
$$\frac{2\sqrt{2}}{3}$$

Note that we can simplify the calculation by making use of the fact that we have symmetry about the y-axis:

$$A = 2\int_0^{\frac{\sqrt{2}}{2}} 1 - 2x^2 dx = 2\left(x - \frac{2}{3}x^3\right)\Big|_0^{\frac{\sqrt{2}}{2}} =$$

$$2\left(\frac{\sqrt{2}}{2} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right) = \sqrt{2}\left(1 - \frac{1}{3}\right) = \frac{2\sqrt{2}}{3}$$

Problem: Find the area of the simple regions lying between the intersections of the curves $y = \sin x$ and $y = \cos x$



We have to be very careful to make sure that the function we take for f lies above the function g on the interval $[a, b]$. We let $a = \frac{\pi}{4}$, $b = \frac{5\pi}{4}$, $f(x) = \sin x$, and $g(x) = \cos x$, so that

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\sin x - \cos x] dx = (-\sin x - \cos x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} =$$

$$\left(-\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) - \left(-\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) =$$

$$\left(-\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

Changing Perspective: functions of y

Suppose we have two functions of y like $f(y) = |y|$ and $g(y) = y^2$ which intersect at c and d, (-1 and 1 in this example) and wish to find the area between them.

We use the formula

$$A = \int_c^d [f(y) - g(y)] dy$$

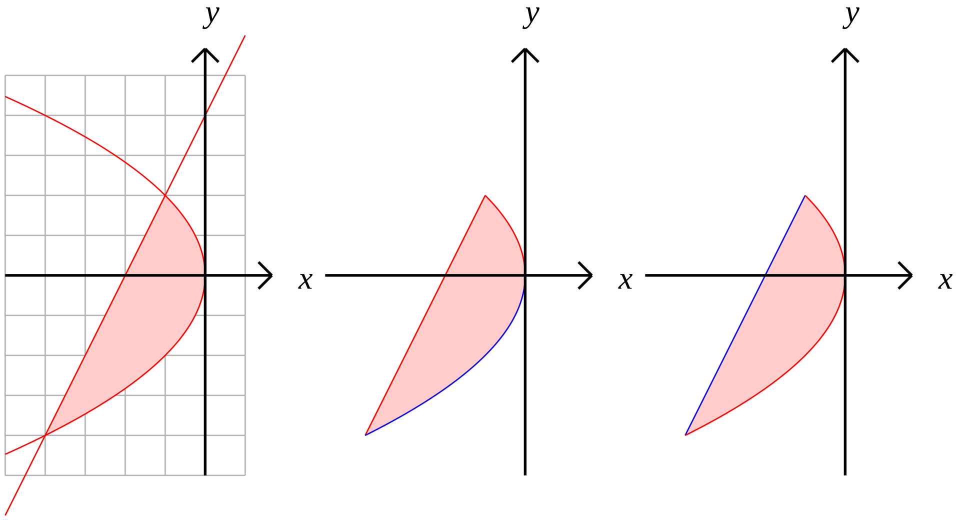
In our example we have

$$A = \int_{-1}^{1} [|y| - y^2] dy = 2\int_0^1 [|y| - y^2] dy = 2\int_0^1 [y - y^2] dy =$$

$$2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

Example: Find the area of the region bounded by the given curves by two methods: (a) integrating with respect to x, (b) integrating with respect to y, if:

$4x + y^2 = 0,\ y = 2x + 4$



Solution: (a) The upper boundary of the region is the graph of the somewhat complicated function

$$f(x) = \begin{cases} 2x+4 & \text{if } -4 \le x \le -1 \\ \sqrt{-4x} & \text{if } -1 \le x \le 0 \end{cases}$$

while the lower part is the graph of $y = -\sqrt{-4x},\ -4 \le x \le 0$.

The area is $A = \int_{-4}^{0} [f(x) - g(x)]\,dx =$

$$\int_{-4}^{-1} [f(x) - g(x)]\,dx + \int_{-1}^{0} [f(x) - g(x)]\,dx =$$

$$\int_{-4}^{-1} \left[2x + 4 - (-\sqrt{-4x})\right] dx + \int_{-1}^{0} \left[\sqrt{-4x} - (-\sqrt{-4x})\right] dx =$$

$$\int_{-4}^{-1} 2x + 4 + 2(-x)^{\frac{1}{2}} dx + 2\int_{-1}^{0} 2(-x)^{\frac{1}{2}} dx =$$

$$x^2 + 4x\Big|_{-4}^{-1} + 2\int_{-4}^{-1} (-x)^{\frac{1}{2}} dx + 4\int_{-1}^{0} (-x)^{\frac{1}{2}} dx =$$

Sidetrack: We need to find $\int (-x)^{\frac{1}{2}} dx$ by making the substitution $u = -x$, $dx = -du$:

$$\int (-x)^{\frac{1}{2}} dx = \int u^{\frac{1}{2}} (-du) = -\int u^{\frac{1}{2}} du =$$

$$-\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{2}{3}(-x)^{\frac{3}{2}} + C$$

Thus we get

$$A = x^2 + 4x\Big|_{-4}^{-1} + 2\int_{-4}^{-1} (-x)^{\frac{1}{2}} dx + 4\int_{-1}^{0} (-x)^{\frac{1}{2}} dx =$$

$$\left((-1)^2 + 4(-1)\right) - \left((-4)^2 + 4(-4)\right) + 2\left(-\frac{2}{3}(-x)^{\frac{3}{2}}\right)\Big|_{-4}^{-1} + 4\left(-\frac{2}{3}(-x)^{\frac{3}{2}}\right)\Big|_{-1}^{0} =$$

$$(1-4) - (16-16) + \left[-\frac{4}{3}(-(-1))^{\frac{3}{2}} - \left(-\frac{4}{3}(-(-4))^{\frac{3}{2}}\right)\right] + \left[4\frac{2}{3}(-0)^{\frac{3}{2}} - 4\frac{-2}{3}(-(-1))^{\frac{3}{2}}\right] =$$

$$-3 + \left[-\frac{4}{3} + \frac{4}{3}(4)^{\frac{3}{2}}\right] + \left[0 - \frac{-8}{3}\right] = -3 + \left[-\frac{4}{3} + \frac{4}{3}8\right] + \frac{8}{3} = 9$$

(b) We first solve the two equations $4x + y^2 = 0$, and $y = 2x + 4$ for x as a function of y and get

$x = -\frac{y^2}{4}$ and $x = \frac{y-4}{2}$

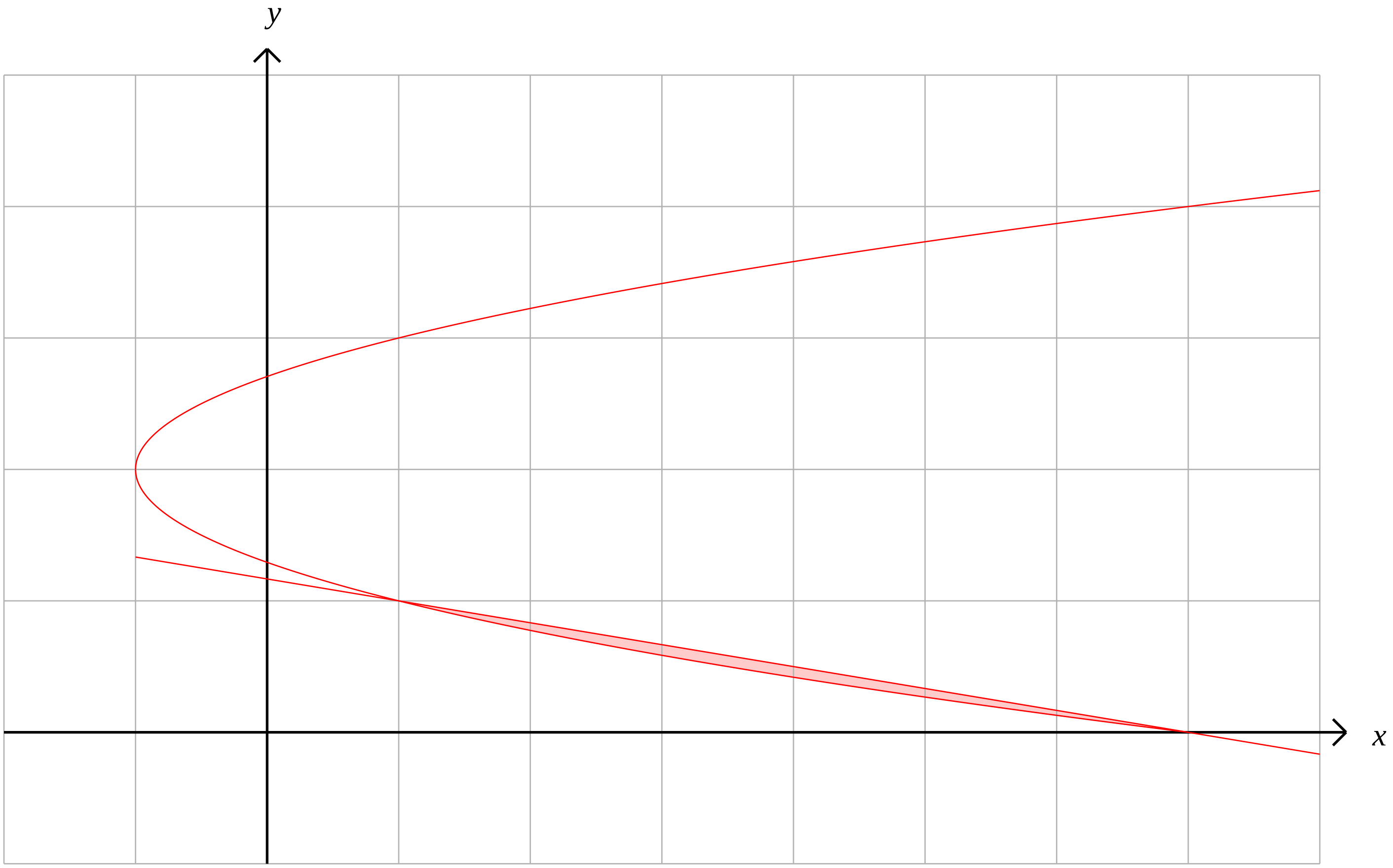
Thus we have $A = \int_{-4}^{2} \left[-\frac{y^2}{4} - \frac{y-4}{2}\right] dy = \int_{-4}^{2} -\frac{y^2}{4} - \frac{y}{2} + 2dy =$

$$-\frac{y^3}{12} - \frac{y^2}{4} + 2y\Big|_{-4}^{2} = \left(-\frac{2^3}{12} - \frac{2^2}{4} + 2(2)\right) - \left(-\frac{(-4)^3}{12} - \frac{(-4)^2}{4} + 2(-4)\right)$$

$$\left(-\frac{8}{12} - \frac{4}{4} + 4\right) - \left(\frac{-64}{12} - \frac{16}{4} - 8\right) = \left(-\frac{2}{3} - 1 + 4\right) - \left(-\frac{-16}{3} - 4 - 8\right) =$$

$$-\frac{2}{3} + 3 - \frac{16}{3} + 12 = 15 - \frac{18}{3} = 9$$

Example: Find the area of the region bounded by the given curves by two methods: (a) integrating with respect to x, (b) integrating with respect to y, if:
$x+1=2(y-2)^2$, $x+6y=7$



Solution: (a) The two curves intersect at the points $(1,1)$ and $(7,0)$, so we have

$$A=\int_1^7\left[\frac{7-x}{6}-\left(2-\sqrt{\frac{x+1}{2}}\right)\right]dx=\int_1^7-\frac{5}{6}-\frac{x}{6}+\sqrt{\frac{x+1}{2}}dx=$$

$$-\frac{5}{6}x-\frac{x^2}{12}+\frac{2}{3\sqrt{2}}(x+1)^{\frac{3}{2}}\Bigg|_1^7=$$

$$\left(-\frac{5}{6}7-\frac{7^2}{12}+\frac{2}{3\sqrt{2}}(7+1)^{\frac{3}{2}}\right)-\left(-\frac{5}{6}1-\frac{1^2}{12}+\frac{2}{3\sqrt{2}}(1+1)^{\frac{3}{2}}\right)=$$

$$\left(-\frac{35}{6}-\frac{49}{12}+\frac{2}{3\sqrt{2}}(8)^{\frac{3}{2}}\right)-\left(-\frac{5}{6}-\frac{1}{12}+\frac{2}{3\sqrt{2}}(2)^{\frac{3}{2}}\right)=$$

$$\left(-\frac{119}{12}+\frac{2}{3\sqrt{2}}8\sqrt{8}\right)-\left(-\frac{11}{12}+\frac{2}{3\sqrt{2}}2\sqrt{2}\right)=$$

$$\left(-\frac{108}{12}+\frac{2}{3\sqrt{2}}8\left(2\sqrt{2}\right)\right)-\left(\frac{4}{3}\right)=-9+\frac{32}{3}-\frac{4}{3}=\frac{1}{3}$$

(b) $A = \int_0^1 \left[(7-6y) - \left(2(y-2)^2 - 1\right)\right] dy =$

$\int_0^1 7 - 6y - \left(2(y^2 - 4y + 4) - 1\right) dy =$

$\int_0^1 7 - 6y - \left(2y^2 - 8y + 8 - 1\right) dy =$

$\int_0^1 -2y^2 + 2y dy = -2\frac{y^3}{3} + 2\frac{y^2}{2}\Big|_0^1 = -\frac{2}{3} + 1 = \frac{1}{3}$

Two strategies become clear from looking at these two examples:

First: if possible, avoid functions whose definitions must involve different formulas on different intervals.

Second: choose the integral that will have the simplest expression.

In both of the examples just looked at, it was best to integrate with respect to y. It is easy to find examples where it is better to integrate with respect to x: just rotate the above examples by 90 degrees!

Volumes of Solid Objects

Let ℓ be a straight coordinate line in space with origin **O**. We denote by $\mathcal{P}_t$ the plane perpendicular to ℓ at a distance t from **O**.

Suppose we have a solid object S which lies between $\mathcal{P}_a$ and $\mathcal{P}_b$ ($a \leq b$), and the area of the cross-section $S \cap \mathcal{P}_t$ of S with $\mathcal{P}_t$ is given by $A(t)$.

Let $V(x)$ be the volume of the solid lying between $\mathcal{P}_a$ and $\mathcal{P}_x$.

We calculate the derivative of $V(x)$:

$$V'(x) = \lim_{h\to 0} \frac{V(x+h) - V(x)}{h} = \lim_{h\to 0} \frac{A(x^*(h))h}{h} = \lim_{h\to 0} \frac{A(x^*(h))h}{h} =$$

$$\lim_{h\to 0} A(x^*(h)) = A(x)$$

where $x^*(h)$ is defined to be the least number x in $[x, x+h]$ for which $V(x+h) - V(x) = A(x)h$.

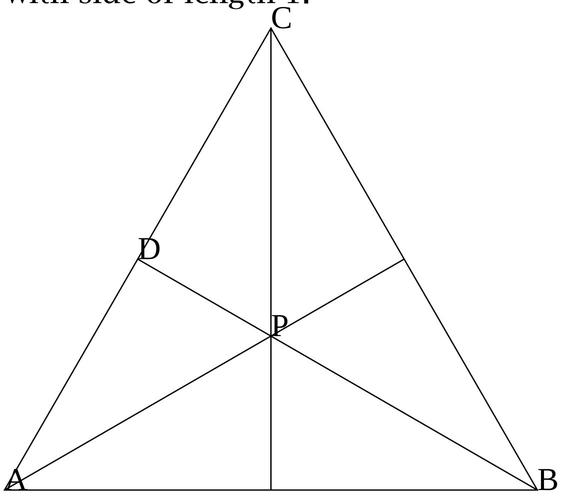
Thus $V(x)$ is an antiderivative of $A(x)$.

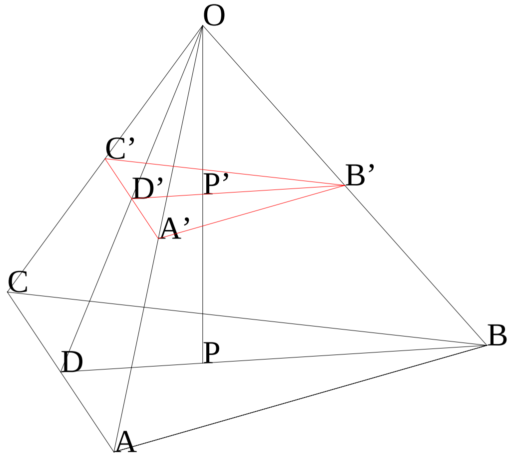
From this we get a basic formula derived from what is called the **Method of Disks** or **Slicing**:

$$V = \int_a^b A(x)dx$$

We usually can count on the cross-sections being simple geometrical figures, such as circles, annulli, squares or standard triangles.

Example: Find the volume of the pyramid S which has 4 faces, all of which are equilateral triangles with side of length 1.





Note: the height of an equilateral triangle whose side is of length 1 is $BD = \frac{\sqrt{3}}{2}$ and its area is thus $\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$. We also have $PD = \frac{1}{3}BD = \frac{1}{3}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6}$

Solution: Let **O** be one of the vertices, and let ℓ be the line through **O** which is perpendicular to the opposite face which it intersects at P. Then

$$OP = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{3}{4} - \frac{3}{36}} = \sqrt{\frac{27}{36} - \frac{3}{36}} = \sqrt{\frac{24}{36}} = \sqrt{\frac{2}{3}}.$$

We let $\mathcal{P}_x$ be the plane perpendicular to ℓ at a distance x from O, intersecting ℓ at P' so that $OP = x$.

Then $\mathcal{P}_x \cap S$ is an equilateral triangle $A'B'C'$ with sides of length $\sqrt{\frac{3}{2}}x$ and area $A(x) = \sqrt{3}\frac{3x^2}{8}$.

The height of the pyramid is $\sqrt{\frac{2}{3}}$, so we evaluate the definite integral

$$V = \int_0^{\sqrt{\frac{2}{3}}} \sqrt{3}\frac{3x^2}{8}dx = \frac{3\sqrt{3}}{8}\int_0^{\sqrt{\frac{2}{3}}} x^2dx = \frac{3\sqrt{3}}{8}\frac{x^3}{3}\Bigg|_0^{\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}}{8}\left(\sqrt{\frac{2}{3}}\right)^3 = \frac{\sqrt{3}}{8}\frac{2}{3}\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{12}$$

which we note is equal to one-third the area $\frac{\sqrt{3}}{4}$ of the base times the height $\sqrt{\frac{2}{3}}$.

Annular Cross-Sections

Definition: An **annulus** is the region lying between two concentric circles. If the radii of the outer and inner circles are $\O$ and $\mathcal{I}$, then the area of the annulus is

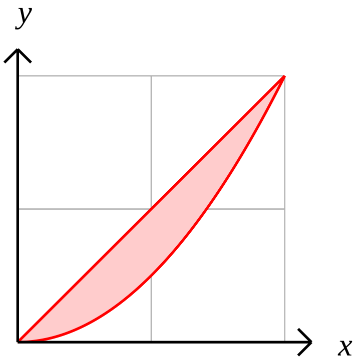
$$A = \pi \O^2 - \pi \mathcal{I}^2 = \pi(\O^2 - \mathcal{I}^2) = \pi(\O + \mathcal{I})(\O - \mathcal{I})$$

It is frequently necessary to compute the volume of an object that has been turned on a lathe. We take ℓ to be the axis of the lathe, and let the outer and inner radii at a distance x from the chuck be $\O(x)$ and $\mathcal{I}(x)$.

Then the volume of the turned object from $x = a$ to $x = b$ is

$$V = \int_a^b \pi \left[\O(x)^2 - \mathcal{I}(x)^2\right] dx = \pi \int_a^b [\O(x) + \mathcal{I}(x)]\,[\O(x) - \mathcal{I}(x)]dx$$

Example: The region $\mathcal{R} = \left\{(x, y) | 0 \le x \le 1,\ \text{and } x^2 \le y \le x\right\}$ is to be rotated about the line $y = -5$. What is the volume of the resulting solid?



Solution: We have $\O(x) = x - (-5) = x + 5$, and $\mathcal{I}(x) = x^2 - (-5) = x^2 + 5$, so

$$V = \pi \int_0^1 \left[(x+5) + (x^2+5)\right]\left[(x+5) - (x^2+5)\right] dx =$$
$$-\pi \int_0^1 \left[x^2 + x + 10\right]\left[x^2 - x\right] dx = -\pi \int_0^1 x^4 - x^2 + 10x^2 - 10x dx =$$
$$-\pi \int_0^1 x^4 + 9x^2 - 10x dx = -\pi \left(\frac{x^5}{5} + 9\frac{x^3}{3} - 10\frac{x^2}{2}\right)\Bigg|_0^1 = \frac{9\pi}{5}$$

Example: The same region is to be rotated about the line $x = -5$. Find the volume generated.

We have to be very careful with this one, because we are dealing with a vertical axis of rotation.

We have to rewrite the region in terms of functions of y:

$\mathcal{R} = \{(x, y)|0 \le y \le 1, \text{ and } y \le x \le \sqrt{y}\}$.

We write the outer and inner radii as functions of y:

$\mathcal{O}(y) = \sqrt{y} - (-5) = y^{\frac{1}{2}} + 5, \mathcal{I}(y) = y - (-5) = y + 5$.

We have $V = \pi \int_0^1 [\mathcal{O}(y) + \mathcal{I}(y)][\mathcal{O}(y) - \mathcal{I}(y)]dy =$

$\pi \int_0^1 [(y^{\frac{1}{2}} + 5) + (y + 5)][(y^{\frac{1}{2}} + 5) - (y + 5)]dy =$

$\pi \int_0^1 [y^{\frac{1}{2}} + y + 10][y^{\frac{1}{2}} - y]dy = \pi \int_0^1 y - y^2 + 10[y^{\frac{1}{2}} - y]dy =$

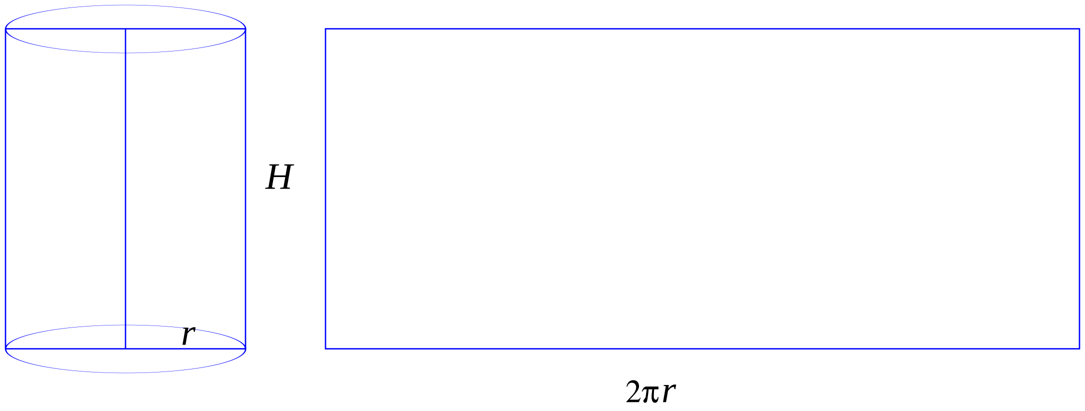
$\pi \int_0^1 -y^2 - 9y + 10y^{\frac{1}{2}}dy = \pi \left(-\frac{y^3}{3} - 9\frac{y^2}{2} + 10\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_0^1 = \pi\left(-\frac{1}{3} - \frac{9}{2} + \frac{20}{3}\right) = \pi\left(\frac{38}{6} - \frac{27}{6}\right) =$
$\frac{11\pi}{6}$

We note that the difference between the two volumes just computed is $\frac{11\pi}{6} - \frac{9\pi}{5} = \pi\frac{55 - 54}{30} = \frac{\pi}{30}$, which is small but possibly important!
It's about 2%.

Volumes by Cylindrical Shells

A **cylindrical shell** is a region contained between two cylinders of the same height with the same central axis. We usually denote the height of the cylinders by H, the radius of the inner cylinder by r, and the thickness of the shell by t, so that the radius of the larger cylinder is $r + t$. The volume of the shell is then the difference between the volumes of the two cylinders:

$$V = \pi(r+t)^2H - \pi r^2H = \pi[(r+t)^2 - r^2]H = \pi(2r+t)tH$$



Notice that if t is very small, then $V = 2\pi rtH + \pi t^2H \doteq 2\pi rtH$

Problem: Suppose we rotate a region
$\mathcal{R} = \{(x,y)|a \le x \le b, \text{ and } g(x) \le y \le f(x)\}$
about the line $x = k$ which lies to the left of $\mathcal{R}$. What is the volume of the resulting solid?

Solution: Let $\mathcal{R}_c$ be the region $\mathcal{R}_c = \{(x,y)|a \le x \le c, \text{ and } g(x) \le y \le f(x)\}$

Then the solid obtained by rotating $\mathcal{R}_{c+h} - \mathcal{R}_c$ about the line $x = k$ is a "cylindrical shell with uneven top and bottom lips" with inner radius $c - k$, outer radius $c + h - k$, thickness h, and height about $f(c) - g(c)$ (if f and g are continous at c.) Thus its volume is about $2\pi(c - k)(f(c) - g(c))h$: think of it as sliced along a vertical seam and flattened into a rectangular solid of thickness h, length equal to the circumference $2\pi(c - k)$ of the shell, and height $f(c) - g(c)$ the same as that of the shell.

Let $V(x)$ be the volume of the solid obtained by rotating $\mathcal{R}_x$ about the line $x = k$.

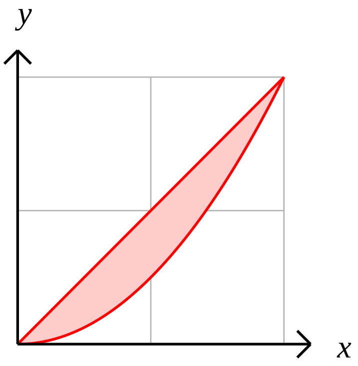
We calculate the derivative of $V(x)$:

$V'(x) = \lim_{h\to 0}\frac{V(x+h)-V(x)}{h} = \lim_{h\to 0}\frac{2\pi(x^*(h)-k)[f(x^*(h))-g(x^*(h))]h}{h} =$
$\lim_{h\to 0}2\pi(x^*(h)-k)[f(x^*(h))-g(x^*(h))] = 2\pi(x-k)[f(x)-g(x)]$
where $x^*(h)$ is defined to be the least number $x$in $[x, x+h]$ for which $V(x+h)-V(x) = 2\pi(x-k)[f(x)-g(x)]h$.
Thus $V(x)$ is an antiderivative of $2\pi(x-k)[f(x)-g(x)]$.

From this we get a basic formula derived from what is called the **Method of Cylindrical Shells**:

$$V = 2\pi\int_a^b (x-k)[f(x)-g(x)]dx \qquad \text{(Vertical Axis Formula)}$$

Example: The region $\mathcal{R} = \left\{(x,y)|0\le x\le 1, \text{ and } x^2\le y\le x\right\}$ is to be rotated about the line $x=-5$. What is the volume of the resulting solid?



Solution: We have $k=-5$, $f(x)=x$, and $g(x)=x^2$, so

$$V = 2\pi\int_0^1 (x-(-5))(x-x^2)dx = 2\pi\int_0^1 (x+5)(x-x^2)dx =$$
$$2\pi\int_0^1 x^2-x^3+5x-5x^2dx = 2\pi\int_0^1 -x^3-4x^2+5xdx =$$
$$2\pi\left(-\frac{x^4}{4}-4\frac{x^3}{3}+5\frac{x^2}{2}\right)\Bigg|_0^1 = 2\pi\left(-\frac{1}{4}-\frac{4}{3}+\frac{5}{2}\right) = \frac{11\pi}{6},$$

which agrees with our previous calculation.

Horizontal Axis of Rotation

Problem: Suppose the region is described in terms of functions of y:
$\mathcal{R} = \{(x, y) | c \le y \le d,\ \text{and}\ g(y) \le x \le f(y)\}$
and the axis of rotation is the line $y = k$ which lies below $\mathcal{R}$. The volume of the resulting solid is given by

$$V = 2\pi \int_c^d (y - k)[f(y) - g(y)]dy \qquad \text{(Horizontal Axis Formula)}$$

Example: The same region is to be rotated about the line $y = -5$. Find the volume generated.

We again rewrite the region in terms of functions of y:

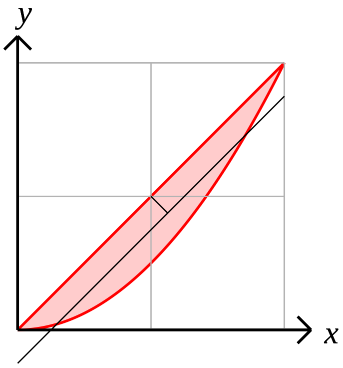
$\mathcal{R} = \left\{(x, y) | 0 \le y \le 1,\ \text{and}\ y \le x \le y^{\frac{1}{2}}\right\}$.

We have $V = 2\pi \int_0^1 (y + 5)(y^{\frac{1}{2}} - y)dy =$
$2\pi \int_0^1 -y^2 + y^{\frac{3}{2}} - 5y + 5y^{\frac{1}{2}} dy =$
$2\pi \left(-\frac{y^3}{3} + \frac{y^{\frac{5}{2}}}{\frac{5}{2}} - 5\frac{y^2}{2} + 5\frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right) \Bigg|_0^1 = 2\pi \left(-\frac{1}{3} + \frac{1}{\frac{5}{2}} - 5\frac{1}{2} + 5\frac{1}{\frac{3}{2}} \right) =$
$2\pi \left(3 + \frac{4}{10} - \frac{25}{10} \right) = 2\pi \left(3 - \frac{21}{10} \right) = \frac{9\pi}{5}$

which again agrees with our previous calculation.

General Rule: The volume equals the definite integral of 2π times the radius and height of the cylinders whose axis is the axis of rotation.

Example: The region $\mathcal{R} = \left\{(x, y) | 0 \le x \le 1, \text{ and } x^2 \le y \le x\right\}$ is to be rotated about the line $y = x$. What is the volume of the resulting solid?



Solution: The lines $y = x - c$ intersect the curve $y = x^2$ when $x^2 = x - c$ or $x^2 - x + c = 0$.

This has solution $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(c)}}{2(1)} = \dfrac{1 \pm \sqrt{1 - 4c}}{2}$ when $1 - 4c \ge 0$ or $c \le \dfrac{1}{4}$.

The lines will only intersect $\mathcal{R}$ when $0 \le c \le \dfrac{1}{4}$.

The distance from the line $y = x$ to the line $y = x - c$ is $\dfrac{\sqrt{2}}{2}c$ and the distance between the two points $\sqrt{2}\sqrt{1 - 4c}$, so the radius and height of the cylinder obtained by rotating the intersection line segment about the line $y = x$ are $\dfrac{\sqrt{2}}{2}c$ and $\sqrt{2}\sqrt{1 - 4c}$, so the volume of the solid is:

$$V = 2\pi \int_0^{\frac{1}{4}} \frac{\sqrt{2}}{2} c \sqrt{2}\sqrt{1 - 4c}\, dc = 2\pi \int_{c=0}^{c=\frac{1}{4}} c\sqrt{1 - 4c}\, dc$$

Letting $u = 1 - 4c$, we get $c = \dfrac{1 - u}{4}$ and $dc = -\dfrac{1}{4}du$. Also $u = 1$ when $c = 0$, and $u = 0$ when $c = \dfrac{1}{4}$.

Thus

$$V = 2\pi \int_{u=1}^{u=0} \frac{1 - u}{4} u^{\frac{1}{2}} \left(-\frac{1}{4}du\right) = \frac{\pi}{8} \int_{u=0}^{u=1} (1 - u) u^{\frac{1}{2}} du =$$

$$V = \frac{\pi}{8} \int_{u=0}^{u=1} u^{\frac{1}{2}} - u^{\frac{3}{2}} du =$$

$$\frac{\pi}{8} \left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}} \right) \Bigg|_{u=0}^{u=1} = \frac{\pi}{8} \left(\frac{2}{3} - \frac{2}{5} \right) = \frac{\pi}{4} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{\pi}{4} \frac{2}{15} = \frac{\pi}{30}$$

Work Done by a Variable Force

We know from basic Science the the work done by a constant force F exerted over a distance d is $W = Fd$. Suppose the force is variable: we might be looking at the displacement of a spring over a distance d, the sending of a rocket into space, or the winding of a cable onto a drum.

If the interval $[a,b]$ is partitioned by a sequence of numbers $x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n$ where $a = x_0$ and $x_n = b$ with $\Delta x_j = x_j - x_{j-1}$ of I_n, and we have a tagset $\mathsf{T} = \{t_1 \le t_2 \le \cdots \le t_n\}$, then the work W done in exerting the force $F(x)$ as x increases from a to b can be approximated by a Riemann sum: $W \doteq \sum_{i=1}^{n} F(t_i)\Delta x_i$

If we take the limit as the mesh of the partition approaches 0, we get a definite integral, which we define to be the work W:

$$W = \lim_{||\mathcal{P}||\to 0} \sum_{i=1}^{n} F(t_i)\Delta x_i = \int_a^b F(x)\,dx$$

Hooke's Law

The force exerted by a spring is directly proportional to its displacement x from its natural length.

$$F(x) = kx\frac{\text{Newton}}{\text{cm}} = kx\frac{\text{N}}{\text{cm}},$$

where $k > 0$ is called the **spring constant** and x is the displacement from the natural length.

Example: A spring has natural length 30 cm, and a force of 50 Newtons(N) is required to extend it to 35 cm. what is the spring constant? How much work is required to extend it from 32 to 36 cm?

Solution: We have $F(35-30) = 50N = k(35-30)\ \text{cm}\frac{\text{N}}{\text{cm}} = k5\text{cm}$, so $k = \frac{50}{5} = 10$

The work required to extend the spring from 32 to 36 cm is

$$W = \int_{(32-30)\text{cm}}^{(36-30)\text{cm}} 10x\frac{\text{N}}{\text{cm}}dx = 10\frac{\text{N}}{\text{cm}}\int_{2\text{cm}}^{6\text{cm}} x\,dx = 10\frac{\text{N}}{\text{cm}}\left(\frac{x^2}{2}\right)\Bigg|_{2\text{cm}}^{6\text{cm}} =$$

$$5\frac{\text{N}}{\text{cm}}\left((6\text{cm})^2 - (2\text{cm})^2\right) = 160\text{N}\cdot\text{cm}$$

Example 4: A conical water tank of height 8m and diameter 6m at the base must have all of its fluid contents pumped to the top of the tank. If it is full to a depth of 4m, how much work will this require?

Solution: Let the interval $[0, 4]$ be partitioned by
$0 = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = 4$, and a tagset $\mathsf{T} = \{t_1 \leq t_2 \leq \cdots \leq t_n\}$ given.

Then the work W_i required to pump the water which lies between level x_{i-1} and x_i approximately equals the force F_i required to lift it times the distance $d_i = 8 - t_i$ that it must be lifted.

The volume V_i of the water lying between level x_{i-1} and x_i is approximately that of the disk of radius $r_i = (8 - t_i)\frac{3}{8}$ and thickness $\Delta x_i = x_i - x_{i-1}$, so $V_i \doteq \pi r_i^2 \Delta x_i = \frac{9\pi}{64}(8 - t_i)^2 \Delta x_i$.

The mass M_i of this volume of water is

$M_i = 1000 \text{ kg } \times V_i \doteq 1000 \times \frac{9\pi}{64}(8 - t_i)^2 \Delta x_i \text{kg}$,

so the force F_i needed to lift it is

$F_i = 9.8 M_i \doteq 9800 \frac{9\pi}{64}\left((8 - t_i)\right)^2 \Delta x_i \text{ N}$.

and therefore the work W_i needed to lift this mass to the top of the tank is about

$F_i d_i = 9800 \frac{9\pi}{64}\left((8 - t_i)\right)^2 \Delta x_i \text{ N}(8 - t_i) = 9800\pi(8 - t_i)^3 \frac{9}{64} \Delta x_i$

Adding them all up, we get an estimate for the total amount of work needed:

$$W \doteq \sum_{i=1}^{n} F_i d_i = \sum_{i=1}^{n} 9800 \frac{9\pi}{64}(8 - t_i)^3 \Delta x_i$$

Taking the limit as the mesh of the partition approaches 0, we get

$$W = \lim_{||\mathcal{P}|| \to 0} \sum_{i=1}^{n} F_i d_i = \sum_{i=1}^{n} 9800 \frac{9\pi}{64}(8 - t_i)^3 \Delta x_i =$$

$$9800 \frac{9\pi}{64} \int_{x=0}^{x=4} (8 - x)^3 dx$$

We use the substitution $u = 8 - x$ to evaluate the definite integral: $dx = -du$, $u = 8$ when $x = 0$, and $u = 4$ when $x = 4$, so

$$9800 \frac{9\pi}{64} \int_{x=0}^{x=4} (8 - x)^3 dx = -9800 \frac{9\pi}{64} \int_{u=8}^{u=4} u^3 du = -9800 \frac{9\pi}{64} \frac{u^4}{4} \Bigg|_{u=8}^{u=4} =$$

$$-9800 \frac{9\pi}{4(64)} 4^4 \left(1^4 - 4^4\right) = -9800(9\pi)(-255) \doteq 70.8 \times 10^6 \text{J}$$

Example: A 40 metre long cable that weighs 5 kg/metre is hanging from the roof of a very tall building. How much work is required to lift it all to the roof level?

Solution: Suppose that x metres of cable has already been lifted to roof level, so that $40 - x$ metres weighing $5(40 - x)$ kg and requiring a force of $9.8 \times 5(40 - x) = 49(40 - x)$ N is left hanging. The work required to lift this remaining cable a distance Δx is therefore $49(40 - x)\Delta x$ J. Summing over a partition of the interval [0,40] gives the Riemann sum

$W \doteq \sum_{i=1}^{n} 49(40 - x)\Delta x \to \int_0^{40} 49(40 - x)dx$ as the mesh of the partition tends to 0.

Evaluating the definite integral, we get

$$W = \int_0^{40} 49(40 - x)dxJ = 49\left(40x - \frac{x^2}{2}\right)\Bigg|_0^{40} J = 49\left(40(40) - \frac{(40)^2}{2}\right)J =$$

$$49(800)J = 39,200J$$