THE VALUATIVE TREE

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ABSTRACT. We describe the set $\mathcal{V}$ of all $\mathbf{R} \cup \{\infty\}$ valued valuations ν on the ring $\mathbf{C}[[x, y]]$ normalized by $\min\{\nu(x), \nu(y)\} = 1$. We show it has a natural structure of $\mathbf{R}$-tree, induced by the order relation $\nu_1 \leq \nu_2$ iff $\nu_1(\phi) \leq \nu_2(\phi)$ for all ϕ. The space $\mathcal{V}$ can also be metrized, endowing it with a metric tree structure. We explain how to obtain these structures directly from the Riemann-Zariski variety of $\mathbf{C}[[x, y]]$. The tree structure on $\mathcal{V}$ also provides an identification of valuations with balls of irreducible curves in a natural ultrametric. Finally, we show that the dual graphs of all sequences of blow-ups patch together, yielding an $\mathbf{R}$-tree naturally isomorphic to $\mathcal{V}$.

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Introduction

In this paper we study valuations on the regular local complete ring $R = \mathbf{C}[[x,y]]$ of formal power series in two variables. Our study is motivated by local, two-dimensional, questions in dynamical systems and complex analysis and even though the present work makes no explicit reference to any of these fields, some of our findings bear their mark.

The earliest systematic study of valuations in two dimensions was done in the fundamental work by Zariski [Z], who, among other things, identified the set $\mathcal{V}_K$ of (not necessarily real valued) valuations on R, vanishing on $\mathbf{C}^*$ and positive on the maximal ideal $\mathfrak{m} = (x,y)$, with sequences of infinitely nearby points. The space $\mathcal{V}_K$ carries a natural topology (the Zariski topology) and is known as the Riemann-Zariski variety. It is a non-Hausdorff quasi-compact space. The obstruction for $\mathcal{V}_K$ being Hausdorff comes from the fact that certain valuations, the *divisorial* ones, do not define closed points. Namely, their associated valuation rings strictly contain other valuation rings. As a remedy to this, we may consider only valuations whose valuation rings are maximal under inclusion. This produces a new space $\mathcal{V}$ which is now Hausdorff and compact. We call it *valuation space*; our aim is to describe its structure in detail.

We identify the elements of $\mathcal{V}$ with valuations $\nu : R \to \overline{\mathbf{R}}_+ = [0,\infty]$, normalized by $\nu(\mathfrak{m}) = 1$ (see Theorem 11.6). The quotient topology on $\mathcal{V}$ induced from $\mathcal{V}_K$ is the topology of pointwise convergence: $\nu_n \to \nu$ iff $\nu_n(\phi) \to \nu(\phi)$ for all $\phi \in R$. We call it the *weak topology.* A *strong* topology can similarly be defined by (normalized) uniform convergence. This topology is metrizable.

Our main result is that these two topologies endow $\mathcal{V}$ with the structure of a tree modeled on the real line. Thus we also refer to $\mathcal{V}$ as the *valuative tree.* We distinguish between two different types of tree structures. A *tree* is a partially ordered set having a unique minimal element (its root) in which segments are isomorphic to real intervals. A *metric tree* is a metric space in which two points are joined by a unique path isometric to a real interval (a metric tree is often called $\mathbf{R}$-tree in the literature.)

The non-metric tree structure arises as follows. For $\nu_1, \nu_2 \in \mathcal{V}$, we say $\nu_1 \le \nu_2$ when $\nu_1(\phi) \le \nu_2(\phi)$ for all $\phi \in R$. The valuation $\nu_{\mathfrak{m}}$ sending ϕ to its multiplicity $m(\phi)$ at the origin is then dominated by any other valuation. This natural order defines a (non-metric) tree structure on $\mathcal{V}$, rooted at $\nu_{\mathfrak{m}}$ (Theorem 7.1). An interesting consequence of the tree structure is that any subset of valuations in $\mathcal{V}$ admits an infimum. Moreover, any tree carries a natural weak tree topology. On $\mathcal{V}$ it coincides with the weak topology (Theorem 9.1).

As for the metric tree structure, any irreducible $\phi \in \mathfrak{m}$ defines an irreducible local formal curve as well as a *curve valuation* $\nu_\phi \in \mathcal{V}$. A curve valuation is a maximal element under $\le$ and the segment $[\nu_{\mathfrak{m}}, \nu_\phi]$ is isomorphic, as a totally ordered set, to the interval $[1,\infty]$. We construct a natural parameterization $[1,\infty] \ni \alpha \mapsto \nu_{\phi,\alpha} \in [\nu_{\mathfrak{m}}, \nu_\phi]$ which is invariant in the following sense: if $f : R \to R$ is a ring automorphism, then the induced map on $\mathcal{V}$ takes $\nu_{\phi,\alpha}$, to $\nu_{f(\phi),\alpha}$. The number α is hence an invariant of a valuation called its *skewness* [1].

[1]the skewness is the inverse of the volume of a valuation as defined in [ELS] (see remark 8.5)

It can be computed directly through the formula $\alpha(\nu) = \sup_\phi \nu(\phi)/m(\phi)$. We use skewness to define a metric on valuation space $\mathcal{V}$, giving it the structure of a metric tree (Theorem 8.8) and show that this tree metric induces the strong topology (Theorem 10.1).

There are four kinds of points in the valuative tree $\mathcal{V}$. The interior points are valuations that become monomial (i.e. determined by their values on x and y) after a finite sequence of blowups. We call them *toroidal*. They include all divisorial valuations but also all *irrational* valuations such as the monomial valuation defined by $\nu(x) = 1$, $\nu(y) = \sqrt{2}$. The ends of $\mathcal{V}$ are curve valuations and *infinitely singular* valuations, which are valuations with infinite multiplicity.

In going from $\mathcal{V}_K$ to $\mathcal{V}$ we discarded certain valuations, namely the ones with nonmaximal valuation rings. These valuations nevertheless do have a natural place in the valuative tree. They are tree tangent vectors at points corresponding to divisorial valuations (see Theorem 11.7). Geometrically they are curve valuations where the curve is defined by an exceptional divisor.

The valuative tree is a beautiful object which may be viewed in a number of different ways. Each corresponds to a particular interpretation of a valuation, and each gives a new insight into it. Some of them will hopefully lead to generalizations in a broader context. Let us describe three such points of views.

The first way consists of identifying valuations with balls of curves. For any two irreducible curves defined by $\phi_1, \phi_2 \in \mathfrak{m}$, set $d(\phi_1, \phi_2) = m(\phi_1)m(\phi_2)/(\phi_1 \cdot \phi_2)$ where $\phi_1 \cdot \phi_2$ is the intersection multiplicity of ϕ_1 and ϕ_2. It is a nontrivial fact that d defines a ultrametric on the set of all irreducible formal curves $\mathcal{C}$ (c.f. [G]). This fact allows us to associate to $(\mathcal{C}, d)$ a metric tree $\mathcal{T}_\mathcal{C}$ by declaring a point in $\mathcal{T}_\mathcal{C}$ to be a closed ball in $\mathcal{C}$, the partial order given by reverse inclusion of balls, and the metric derived from radii of balls. Theorem 15.2 states that the tree $\mathcal{T}_\mathcal{C}$ is isometric to the valuative tree $\mathcal{V}$ with its end points removed (i.e. to the set $\mathcal{V}_{\rm tor}$ of toroidal valuations). We use this result to extend intersection products and multiplicities to any valuation.

The second way uses Zariski's identification of valuations with sequences of infinitely nearby points. We let Γ_π be the dual graph of a finite sequence of blow-ups π. It is a simplicial tree and as such defines a nonmetric tree. When one sequence π' contains another π, the dual graph Γ_π is naturally a subtree of $\Gamma_{\pi'}$. This allows to take the union Γ of Γ_π over all sequences π of blow-ups above the origin. The result, called the universal dual graph, is a nonmetric tree that we show is isomorphic to the set $\mathcal{V}_{\rm tor}$ of toroidal valuations. Moreover, we may recursively and consistently attach to any vertex in any Γ_π a vector $(a, b) \in (\mathbf{N}^*)^2$, called its Farey weight and the associated Farey parameter a/b (see also [HP]). The Farey parameter induces on $\mathcal{V}_{\rm tor}$ a new invariant of valuation called the *thinness* which is a modified version of skewness using the multiplicity of valuations. We may use the thinness to parameterize $\mathcal{V}_{\rm tor}$: this induces a new metric tree structure on $\mathcal{V}_{\rm tor}$, the *thin* metric. See Theorem C.2.

A third way is to view the valuative tree in the context of Berkovich spaces and Bruhat-Tits buildings. As a nonmetric tree, $\mathcal{V}$ embeds as the closure of a disk in the Berkovich projective line $\mathbb{P}^1(k)$ over the local field $k = \mathbf{C}((x))$. The thin metric on $\mathcal{V}$ then also arises from an identification of a subset of the Berkovich projective line with the Bruhat-Tits building of PGL_2.

The applications of the tree structure of $\mathcal{V}$ to dynamics and analysis will be explored in forthcoming papers [FJ1, FJ2].

We have divided this paper into five parts and an appendix. In the first part we give basic definitions, examples and results on valuations. In particular we describe the relationship between valuations and blow-ups (in dimension two).

Part 2 is the technical heart of the paper. We encode valuations by finite or countable pieces of data that we call sequences of toroidal key polynomials, or *STKP's*. This encoding is an adaptation of a method by MacLane [M] which was indicated to us by B. Teissier. An STKP (or at least a subsequence of it) corresponds to generating polynomials and approximate roots in the language of Spivakovsky [Sp] and Abhyankar-Moh [AM], respectively. We are thus able to classify valuations on R. This classification is well-known to specialists (see [ZS], [Sp] for instance) but we feel that our concrete approach is of independent interest. The representation of valuations by STKP's is the key to the tree structure of valuation space. We note that while the method of STKP's depends on a choice of coordinates (i.e. writing $R = \mathbf{C}[[x]][[y]]$), the resulting tree structure is completely coordinate independent.

The third part, consisting of Sections 5-7, concerns tree structures. Section 5 contains generalities on metric and non-metric trees. In Section 6 we show, using the encoding by STKP's, that $\mathcal{V}$ has a non-metric structure induced by its natural partial ordering. In Section 7, we introduce skewness and use it to define a natural metric tree structure on $\mathcal{V}$. A tree structure was also described in a context similar to ours in [AA], but without any explicit mention to valuations.

In Part 4 we analyze different topologies on $\mathcal{V}$. We start by giving the main properties of the weak (Section 8) and strong (Section 9) topologies on $\mathcal{V}$. In Section 10, we describe the Zariski topology on $\mathcal{V}_K$ and show that it projects onto the weak topology on $\mathcal{V}$. Then we show in Section 11.3 that the "discarded" valuations in $\mathcal{V}_K \setminus \mathcal{V}$ correspond naturally to tangent vectors in the valuative tree. Yet another topology on $\mathcal{V}_K$, and hence on $\mathcal{V}$, is given by the Hausdorff-Zariski topology. It is studied in Section 12 where we show that it is equivalent to the weak tree topology induced by a natural discrete tree structure on $\mathcal{V}_K$. Finally in Section 13 we compare all these topologies on $\mathcal{V}$.

The fifth part makes precise the identification of toroidal valuations with balls of curves. We also extend the concept of the multiplicity of a curve and intersection multiplicity of two curves, to arbitrary valuations. This will be important for our applications to analysis and dynamical systems.

In the appendix we explore the nonmetric tree structure on $\mathcal{V}$ using Puiseux expansions. This approach also gives a metric tree structure defined in terms of thinness. We show how these structures are connected to the Berkovich projective line and Bruhat-Tits building of PGL_2 over the local field $\mathbf{C}((x))$. Finally we build the universal dual tree Γ described above, and show that it is isometric to the set of toroidal valuations endowed with the thin metric.

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Part 1. Generalities

In this part, we give basic results on valuations on a complete regular local ring R of dimension 2 with algebraically closed residue field k of characteristic zero. For definiteness we assume $k = \mathbf{C}$. By choosing regular parameters (i.e. fixing coordinates) we may assume $R = \mathbf{C}[[x, y]]$, the ring of formal power series. Let K be the fraction field of R and $\mathfrak{m} = xR + yR$ the maximal ideal.

We distinguish between *valuations* and *Krull valuations*[2]. Valuations will always be $\mathbf{R} \cup \{\infty\}$ valued, whereas Krull valuations more generally take values in an abstract totally ordered group. This distinction may seem unnatural but is important for our purposes.

We start by giving precise definitions of (Krull) valuations and their associated invariants. Then we describe some important examples and discuss the link between valuations and Krull valuations. Finally we prove Zariski's correspondence between Krull valuations and sequences of blow-ups.

All results of this part are basic, and will be freely used in the sequel. Our two references are [ZS] and [V]; see also [Sp].

1.1. **Valuations.** We set $\overline{\mathbf{R}}_+ := \mathbf{R}_+ \cup \{\infty\}$, extending the addition and multiplication on $\mathbf{R}_+$ to $\overline{\mathbf{R}}_+$ in the usual way.

Definition 1.1. A *valuation* on R is a nonconstant function $\nu : R \to \overline{\mathbf{R}}_+$ with:

(V1) $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ for all $\phi, \psi \in R$;
(V2) $\nu(\phi + \psi) \geq \min\{\nu(\phi), \nu(\psi)\}$ for all $\phi, \psi \in R$;
(V3) $\nu(1) = 0$.

This implies $\nu(0) = \infty$ and $\nu|_{\mathbf{C}^*} = 0$. The set $\mathfrak{p} := \{\phi \in R \mid \nu(\phi) = \infty\}$ is a prime ideal in R; we say ν is *proper* if $\mathfrak{p} \subsetneq \mathfrak{m}$. It is *centered* if it is proper and if $\nu(\mathfrak{m}) := \min\{\nu(\phi) \mid \phi \in \mathfrak{m}\} > 0$. Two centered valuations ν_1, ν_2 are *equivalent*, $\nu_1 \sim \nu_2$, if $\nu_1 = C\nu_2$ for some constant $C > 0$. Any centered valuation ν is equivalent to a unique valuation $\widetilde{\nu}$ *normalized* by $\widetilde{\nu}(\mathfrak{m}) = 1$. We denote by $\tilde{\mathcal{V}}$ the set of all centered valuations and by $\mathcal{V}$ the quotient of $\tilde{\mathcal{V}}$ by the equivalence relation $\sim$, i.e. the set of normalized valuations.

We endow $\tilde{\mathcal{V}}$ and $\mathcal{V}$ with the weak topology: if ν_k, ν are valuations in $\tilde{\mathcal{V}}$ $(\mathcal{V})$, then $\nu_k \rightharpoonup \nu$ in $\tilde{\mathcal{V}}$ $(\mathcal{V})$ iff $\nu_k(\phi) \to \nu(\phi)$ for all $\phi \in R$. As we will see, the weak topology on $\mathcal{V}$ is induced by a tree structure.

Finally $\tilde{\mathcal{V}}$ and $\mathcal{V}$ come with natural partial orderings: $\nu_1 \leq \nu_2$ iff $\nu_1(\phi) \leq \nu_2(\phi)$ for all $\phi \in R$.

1.2. **Krull valuations.** We now introduce the concept of a Krull valuation.

Definition 1.2. Let Γ be a totally ordered group. A *Krull valuation* on R is a function $\nu : R \to \Gamma$ satisfying (V1)-(V3) above.

A Krull valuation is *centered* if $\nu \geq 0$ on R and $\nu > 0$ on $\mathfrak{m}$. Two Krull valuations $\nu_1 : R \to \Gamma_1$, $\nu_2 : R \to \Gamma_2$ are *equivalent* if $h \circ \nu_1 = \nu_2$ for some strictly increasing homomorphism $h : \Gamma_1 \to \Gamma_2$. Any Krull valuation extends naturally to the fraction field K of R by $\nu(\phi/\psi) = \nu(\phi) - \nu(\psi)$.

[2] This is not standard terminology but will be used throughout the paper.

A *valuation ring* V is a local ring with fraction field K such that $x \in K^*$ implies $x \in V$ or $x^{-1} \in V$. Say that $x, y \in V$ are equivalent iff xy^{-1} is a unit in V. The quotient Γ_V is endowed with a natural total order given by $x \geq y$ iff $xy^{-1} \in \mathfrak{m}_V$. The projection $V \to \Gamma_V$ then extends to a Krull valuation $\nu_V : K \to \Gamma_V$.

Conversely, if ν is a centered Krull valuation, then $R_\nu := \{\phi \in K \mid \nu(\phi) \geq 0\}$ is a valuation ring with maximal ideal $\mathfrak{m}_\nu := \{\nu(\phi) > 0\}$. One can show that ν is equivalent to ν_{R_ν} defined above. In particular, two Krull valuations ν, ν' are equivalent iff $R_\nu = R_{\nu'}$.

We let $\tilde{\mathcal{V}}_K$ be the set of all centered Krull valuations, and $\mathcal{V}_K$ be the quotient of $\tilde{\mathcal{V}}_K$ by the equivalence relation. Equivalently, $\mathcal{V}_K$ is the set of valuation rings V in K whose maximal ideal $\mathfrak{m}_V$ satisfies $R \cap \mathfrak{m}_V = \mathfrak{m}$.

A Krull valuation carries several numerical invariants (see e.g. Table 1 in Section 17).

Definition 1.3. Let $\nu : R \to \Gamma$ be a centered Krull valuation.

- The *rank* $\mathrm{rk}(\nu)$ of ν is the Krull dimension of the ring R_ν.
- The *rational rank* of ν is defined by $\mathrm{rat.rk}(\nu) := \dim_{\mathbf{Q}}(\nu(K) \otimes_{\mathbf{Z}} \mathbf{Q})$.
- The *transcendence degree* $\mathrm{tr.deg}(\nu)$ of ν is defined as follows. Since ν is centered, we have a natural inclusion $k := R/\mathfrak{m} \subset k_\nu := R_\nu/\mathfrak{m}_\nu$. We let $\mathrm{tr.deg}(\nu)$ be the transcendence degree of this field extension.

One can show that $\mathrm{rk}(\nu)$ is the least integer l so that $\nu(K)$ can be embedded as an ordered group into $(\mathbf{R}^l, +)$ endowed with the lexicographic order. Hence both $\mathrm{rk}(\nu)$ and $\mathrm{rat.rk}(\nu)$ depend only on the value group $\nu(K)$ of the valuation. We will give a geometric interpretation of $\mathrm{tr.deg}(\nu)$ later (see Remark 1.8).

Abhyankar's inequalities state that:

$$\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \mathrm{rat.rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \dim R = 2. \tag{1.1}$$

If $\mathrm{rat.rk}(\nu) + \mathrm{tr.deg}(\nu) = 2$, then $\nu(K)$ is isomorphic (as a group) to $\mathbf{Z}^\varepsilon$ with $\varepsilon = \mathrm{rat.rk}(\nu)$. When $\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) = 2$, $\nu(K)$ is isomorphic as an *ordered* group to $\mathbf{Z}^\varepsilon$ endowed with the lexicographic order.

To ν we associate a *graded ring* $\mathrm{gr}_\nu R = \oplus_{r \in \Gamma}\{\nu \geq r\}/\{\nu > r\}$. It is in bijection with equivalence classes of R under the equivalence relation $\phi = \psi$ *modulo* ν iff $\nu(\phi - \psi) > \nu(\phi)$. We have $\overline{\phi \cdot \psi} = \overline{\phi} \cdot \overline{\psi}$ in $\mathrm{gr}_\nu R$ for any $\phi, \psi \in R$.

Any ring homomorphism $f : R \to R$, induces actions on $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}_K$ given by $f_*\nu(\phi) = \nu(f(\phi))$. When f is an automorphism f_* is a bijection and preserves all three invariants rk, rat.rk and tr.deg defined above. It also restricts to a bijection on $\mathcal{V}$ preserving the natural partial order: $f_*\nu \leq f_*\nu'$ as soon as $\nu \leq \nu'$.

1.3. **Examples.** Let us describe a few examples and introduce some terminology. See also Definition 3.7.

(E0) *The* $\mathfrak{m}$*-valuation.* As R is a local ring with maximal ideal $\mathfrak{m}$, the function

$$\nu_{\mathfrak{m}}(\phi) := \min\{k \mid \phi \in \mathfrak{m}^k\},$$

defines both a normalized valuation and a Krull valuation. Notice that $\nu_{\mathfrak{m}}(\phi)$ is the multiplicity of ϕ at the origin; we will often write $m(\phi)$ instead of $\nu_{\mathfrak{m}}(\phi)$.

(E1) *Monomial valuations.* Pick $\alpha > 0$, set $\nu(x) = 1$, $\nu(y) = \alpha$ and declare

$$\nu(\phi) := \min\{i + \alpha j \mid a_{ij} \neq 0\},$$

for $\phi = \sum a_{ij} x^i y^j$. This defines a normalized valuation (and a Krull valuation) called a *monomial* valuation. We have $\mathrm{rk}(\nu_\alpha) = 1$ for any α. When $\alpha \in \mathbf{Q}$, $\mathrm{tr.deg}(\nu_\alpha) = 1$ and $\mathrm{rat.rk}(\nu_\alpha) = 1$; when $\alpha \notin \mathbf{Q}$, $\mathrm{tr.deg}(\nu_\alpha) = 0$ and $\mathrm{rat.rk}(\nu_\alpha) = 2$. Of course, $\nu_{\mathfrak{m}}$ is the monomial valuation with $\alpha = 1$.

(E2) *Divisorial valuations.* Consider a finite sequence $\pi_i : X_i \to X_{i-1}$ of blow-ups at points p_{i-1} with exceptional divisors $E_i = \pi_i^{-1}(p_{i-1})$, $1 \leq i \leq n$, $p_0 = (0,0)$. We say that (p_i) is a *sequence of infinitely nearby points* if $p_i \in E_i$. Set $\varpi := \pi_1 \circ \cdots \circ \pi_n$ and $E := E_n$, the last exceptional divisor. For $\phi \in K$ let $\mathrm{div}_E(\phi) \in \mathbf{Z}$ be the order of vanishing of ϕ along E. There is then a unique integer $b = b(E) > 0$ such that $\nu_E(\phi) := b^{-1} \mathrm{div}_E(\pi^*\phi)$ defines a normalized valuation (and a Krull valuation) on R, called a *divisorial* valuation. Its invariants are $\mathrm{rk}(\nu_E) = \mathrm{rat.rk}(\nu_E) = 1$ and $\mathrm{tr.deg}(\nu_E) = 1$; any valuation with these invariants is in fact divisorial (see [ZS]). Any monomial valuation with $\alpha \in \mathbf{Q}$ is divisorial.

(E3) *Toroidal valuations.* Let ϖ, E be as in (E2), pick $p \in E$ and fix local coordinates at p. The image $\nu = \pi_*\nu_\alpha$ of any monomial valuation is a *toroidal* valuation. When $s \in \mathbf{Q}$ it is divisorial. When $s \notin \mathbf{Q}$, we say that ν is an *irrational* valuation. Its invariant are $\mathrm{rk}(\nu) = 1$, $\mathrm{rat.rk}(\nu) = 2$ and $\mathrm{tr.deg}(\nu) = 0$.

(E4) *Curve valuations.* Let $V = \{\phi = 0\}$ be an irreducible formal curve. For $\psi \in R$, write $\psi = \phi^k \hat{\psi}$ with $k \in \mathbf{N}$, $\hat{\psi}$ prime with ϕ, and define

$$\nu_\phi(\psi) = \phi \cdot \psi / m(\phi) \in \mathbf{Q}_+ \cup \{\infty\} \quad \text{and} \quad \nu_V(\psi) := (k, \nu_\phi(\hat{\psi})) \in \mathbf{Z} \times \mathbf{Q}.$$

Here $\phi \cdot \psi$ denotes the intersection multiplicity and $m(\phi)$ the multiplicity as in E1. Then ν_ϕ is a normalized valuation and ν_V a centered Krull valuation, where $\mathbf{Z} \times \mathbf{Q}$ is lexicographically ordered. The invariants of ν_V are $\mathrm{rk}(\nu_V) = \mathrm{rat.rk}(\nu_V) = 2$ and $\mathrm{tr.deg}(\nu_V) = 0$. Such (Krull) valuations are called *curve valuations.*

(E5) *Exceptional curve valuations.* This is an modification of (E4). Let π, E, b and ν_E be as in (E2) and pick $p \in E$. Assume $E = \{u = 0\}$ in local coordinates (u, v) near $p = (0,0)$. For $\phi \in R$ write $\pi^*\phi = u^{b\nu_E(\phi)} \cdot \psi$, where $\psi(0, v) \not\equiv 0$. Then $\psi(0, v) = v^k \hat{\psi}(v)$, where $v \nmid \hat{\psi}$ and

$$\nu_{E,p}(\phi) := (b\nu_E(\phi), k) \in \mathbf{Z} \times \mathbf{Z}$$

defines a centered Krull valuation with invariants $\mathrm{rk}(\nu_{E,p}) = \mathrm{rat.rk}(\nu_{E,p}) = 2$ and $\mathrm{tr.deg}(\nu_{E,p}) = 0$. We call $\nu_{E,p}$ an *exceptional curve valuation.*

The remaining valuations on R are *infinitely singular* valuations. They are defined by $\mathrm{rat.rk} = \mathrm{rk} = 1$ and $\mathrm{tr.deg} = 0$. One can show that their value groups are not finitely generated over $\mathbf{Z}$.

Let us include the following result, which will be used on several occasions.

Lemma 1.4. *Let μ, ν be non-trivial Krull valuations on R with $R \subset R_\mu \subsetneq R_\nu$. Then either $\nu \sim \nu_E$ is divisorial and $\mu = \nu_{E,p}$ is an exceptional curve valuation; or $\nu(\psi) = \max\{k \mid \phi^k | \psi\}$ for some irreducible $\phi \in \mathfrak{m}$ and any $\psi \in R$, and $\mu \sim \nu_{\phi^{-1}(0)}$ is a curve valuation as in (E4).*

Proof. As $R_\mu \subsetneq R_\nu$, $\mathfrak{p} := \mathfrak{m}_\nu \cap R_\mu$ is a prime ideal, strictly included in R_μ, and the quotient ring $R_\mu/\mathfrak{p}$ is a non-trivial valuation ring of the residue field $k_\nu = R_\nu/\mathfrak{m}_\nu$. Conversely, a valuation κ on the residue field k_ν determines a unique valuation μ' on K with $R_{\mu'} \subsetneq R_\nu$, and $R_{\mu'}/\mathfrak{m}_\nu \cap R_{\mu'} = R_\kappa$. One says that μ' is the composite valuation of ν and κ. When the value group of ν is isomorphic to $\mathbf{Z}$, we have $\mu'(\psi) = (\nu(\psi), \kappa(\overline{\psi}))$, where $\overline{\psi}$ is the class of ψ in k_ν (see e.g. [V] p.554).

Suppose ν is centered. By Abhyankar's inequality, $\mathrm{tr.deg}(\nu)$ is either 0 or 1. In the first case, k_ν is isomorphic to $\mathbf{C}$, which admits no nontrivial nonnegative valuation. Hence $\mathrm{tr.deg}(\nu) = 1$, $\nu \sim \nu_E$ is divisorial and the valuation κ defined above has rank 1. We may hence assume $\kappa(K) \subset \mathbf{R}$ and $\mu = (\nu, \kappa) \in \mathbf{R} \times \mathbf{R}$.

Performing the suitable sequence of blow-ups and lifting the situation, one can suppose $\nu = \nu_{\mathfrak{m}}$, the $\mathfrak{m}$-adic valuation. Assume $\nu(y) \geq \nu(x)$, or else switch the roles of x and y. Since $\mu \neq \nu_{\mathfrak{m}}$ there exists a unique $\theta \in \mathbf{C}$ for which $\mu(y - \theta x) > \mu(x)$. Then μ is equivalent to the unique valuation sending x to $(1, 0)$ and $y - \theta x$ to $(1, 1)$. This is the exceptional curve valuation attached to the exceptional curve obtained by blowing-up the origin once and at the point of intersection with the strict transform of $\{y - \theta x = 0\}$.

When ν is not centered, the prime ideal $R \cap \mathfrak{m}_\nu$ is generated by an irreducible element $\phi \in \mathfrak{m}$, and $\nu(\psi) = \max\{k \mid \phi^k | \psi\}$. The residue field k_ν is isomorphic to $\mathbf{C}((t))$, on which there exists a unique valuation vanishing on $\mathbf{C}^*$. Therefore μ is equivalent to $\nu_{\phi^{-1}(0)}$. □

1.4. **Valuations versus Krull valuations.** Consider a centered valuation $\nu : R \to \overline{\mathbf{R}}_+$ and let $\mathfrak{p} = \{\nu = \infty\} \subsetneq \mathfrak{m}$. When $\mathfrak{p} = (0)$, ν defines a Krull valuation whose value group is included in $\mathbf{R}$. In particular $\mathrm{rk}(\nu) = 1$.

When $\mathfrak{p} = (\phi)$ is nontrivial, $\phi \in \mathfrak{m}$ irreducible, then ν is equivalent to the curve valuation ν_ϕ in (E4) by Lemma 1.4. Define $\mathrm{krull}\,[\nu] : R \to \mathbf{Z} \times \mathbf{R}$ to be the Krull valuation ν_V associated to $V = \{\phi = 0\}$ as in (E4).

It is straightforward to check that if two valuations give rise to equivalent Krull valuations, then the valuations are themselves equivalent. In particular we may define the numerical invariants rk, rat.rk and tr.deg for all valuations.

Conversely, pick a Krull valuation ν. When $\mathrm{rk}(\nu) = 1$, its value group can be embedded (as on ordered group) in $(\mathbf{R}, +)$ so that ν defines a valuation. When $\mathrm{rk}(\nu) = 2$, two cases may appear.

(1) There are non-units $\phi_1, \phi_\infty \in R$ with $n\nu(\phi_1) < \nu(\phi_\infty)$ for all $n \geq 0$. Then

$$\mathfrak{p} := \{\phi \mid n\nu(\phi_1) < \nu(\phi) \text{ for all } n \geq 0\}$$

is a proper prime ideal generated by an irreducible element we again denote by ϕ_∞. We define a valuation $\nu_0 : R \to \overline{\mathbf{R}}_+$ as follows. First set $\nu_0(\phi_1) = 1$, and $\nu_0(\phi) = \infty$ for $\phi \in \mathfrak{p}$. For any $\phi \in R \setminus \mathfrak{p}$, let

$$n_k := \max\{n \in \mathbf{N} \mid n \cdot \nu(\phi_1) \leq \nu(\phi^k)\}.$$

One checks that $k^{-1}n_k \leq (kl)^{-1}n_{kl} \leq k^{-1}n_k + k^{-1} - (kl)^{-1}$ for all $k, l \geq 0$. This implies that the sequence n_k/k converges towards a real number we define to be $\nu_0(\phi)$. The function ν_0 is a valuation, and $\mathrm{krull}\,[\nu_0]$ is isomorphic to ν.

(2) For all non-units $\phi, \psi \in R$ there exists $n \geq 0$ so that $n\nu(\psi) > \nu(\phi)$. In this case, there can be no valuation $\nu_0 : R \to \overline{\mathbf{R}}_+$ so that $\mathrm{krull}\,[\nu_0] \sim \nu$. Let us show

that ν is an exceptional curve valuation as in (E5). As $\operatorname{rk}\nu = 2$, we may assume $\nu(R) \subset \mathbf{R} \times \mathbf{R}$, and $\nu = (\nu_0, \nu_1)$. The function $\nu_0 : R \to \mathbf{R}$ is also a valuation, as ν is and $\mathbf{R} \times \mathbf{R}$ is endowed with the lexicographic order. Since $R_\nu \subsetneq R_{\nu_0}$ Lemma 1.4 shows that ν_0 is divisorial and ν an exceptional curve valuation.

We have hence proved

Proposition 1.5. *To any Krull valuation ν not of type (E5) above is associated a unique (up to equivalence) valuation $\nu_0 : R \to \overline{\mathbf{R}}_+$ with* $\operatorname{krull}[\nu_0] \sim \nu$*.*

In other words, the image of the map $\nu_0 \mapsto \operatorname{krull}[\nu_0]$ from the set of valuations $\mathcal{V}$ to the set of Krull valuations $\mathcal{V}_K$ is exactly $\mathcal{V}_K$ with all exceptional curve valuations removed.

1.5. **Sequences of blow-ups and Krull valuations.** Let us now describe the geometrical point of view of valuations as developed by Zariski (see [ZS],[V]).

We start by a naive approach. Let $\nu \in \tilde{\mathcal{V}}_K$ be a centered Krull valuation on $R = \mathbf{C}[[x, y]]$. Assume $\nu(y) \geq \nu(x)$. Then $\nu \geq 0$ on $R[y/x] \subset K$ and $Z_\nu := \{\phi \in R[y/x] \mid \nu(\phi) > 0\}$ is a prime ideal containing x.

When $Z_\nu = (x)$, the value $\nu(\phi)$ for $\phi \in R[y/x]$ is proportional to the multiplicity of ϕ along x. Thus $\nu = \nu_\mathfrak{m}$ (see Example (E0)). Otherwise $Z_\nu = (y/x - \theta, x)$ where $\theta \in \mathbf{C}$. Define $x_1 = x$ and $y_1 = y/x - \theta$. Then ν is a centered Krull valuation on the local ring $\mathbf{C}[[x_1, y_1]] \supset R$. When $\nu(y) < \nu(x)$ we get the same conclusion by replacing x and y. We may now iterate this procedure.

A more geometric formulation of this construction requires some terminology. If $(R_1, \mathfrak{m}_1)$, $(R_2, \mathfrak{m}_2)$ are local rings with common fraction field K, then we say that $(R_1, \mathfrak{m}_1)$ *dominates* $(R_2, \mathfrak{m}_2)$ if $R_1 \supset R_2$ and $\mathfrak{m}_2 = R_2 \cap \mathfrak{m}_1$.

Definition 1.6. Let X be a scheme whose field of rational functions is K and ν a Krull valuation on K with valuation ring R_ν. The *center* of ν in X is the unique point $x \in X$ whose local ring $\mathcal{O}_{X,x} \subset K$ is dominated by R_ν.

Again consider $\nu \in \tilde{\mathcal{V}}_K$. The center of ν in $\operatorname{Spec} R$ is $\mathfrak{m}$. Blow-up the origin (i.e. $\mathfrak{m}$) $\pi_1 : X_1 \to \operatorname{Spec} R$, and let $E_1 = \pi_1^{-1}(0)$ be the exceptional divisor. Consider the center of ν on X_1. Either this is the generic point of E_1, in which case $\nu \sim \nu_\mathfrak{m}$, or a unique closed point $p_1 \in E_1$.

In the latter case we consider the restriction of ν to the local ring $\mathcal{O}_{p_1}$, whose fraction field is K, and repeat the procedure. That is, we blow up p_1 to obtain X_2, and consider the center of ν in X_2. This is either the generic point of the exceptional divisor E_2, in which case $\nu \sim \nu_{E_2}$ is divisorial, or a unique closed point $p_2 \in E_2$. Iterating this, we obtain a sequence $\Pi[\nu] = (p_i)$ of infinitely nearby points above the origin p_0, the length of which is finite iff ν is divisorial.

Conversely, let $\bar{p} = (p_i)_0^n$ be a sequence of infinitely nearby points with associated blowups π_i and exceptional divisors E_i. Write $\varpi_i := \pi_1 \circ \cdots \circ \pi_i$. Define $R_{\bar{p}} \subset K$ as follows. When $n < \infty$, then $\phi \in R_{\bar{p}}$ iff $\varpi_n^* \phi$ is regular at a generic point on E_n. If $n = \infty$, then $\phi \in R_{\bar{p}}$ iff there exists $i \geq 1$ such that $\varpi_i^* \phi$ is regular at p_i (which implies that $\varpi_j^* \phi$ is regular at p_j for $j \geq i$.) A direct check shows that $R_{\bar{p}}$ is a ring, and since any curve can be desingularized by a finite sequence of blow-ups, we get either $\phi \in R_{\bar{p}}$ or $\phi^{-1} \in R_{\bar{p}}$ for any $\phi \in K^*$. Thus $R_{\bar{p}}$ is a valuation ring, and we denote by $\operatorname{val}(\bar{p})$ its associated Krull valuation.

One can show that $R_{\Pi(\nu)} = R_\nu$ so that $\mathrm{val}\,[\Pi(\nu)] \sim \nu$. Conversely, for a fixed sequence $\bar{p}$, the sequence associated to $R_{\bar{p}}$ is again given by $\bar{p}$.

Summing up, we have

Theorem 1.7. *The set $\mathcal{V}_K$ of centered Krull valuations modulo equivalence is in bijection with the set of sequences of infinitely nearby points. The bijection is given as follows: $\nu \in \mathcal{V}_K$ corresponds to (p_i) iff R_ν dominates $\mathcal{O}_{p_i}$ for all i.*

Remark 1.8. The correspondence above is a particular case of a general result (see [ZS]). Moreover, the residue field $k_\nu \cong R_\nu/\mathfrak{m}_\nu$ of a valuation is isomorphic to the field of rational functions of the "last" exceptional divisor of the sequence $\Pi[\nu]$. In our case, either ν is non divisorial and $k_\nu \cong \mathbf{C}$; or ν is divisorial and $k_\nu \cong \mathbf{C}(T)$ for some indeterminate. We will explicitly produce such isomorphisms in Section 3.

Example 1.9. The sequence $\Pi[\nu]$ is finite iff ν is a divisorial valuation. In this case, $\Pi[\nu]$ coincides with the defining sequence of ν. For instance $\Pi[\nu_{\mathfrak{m}}] = (p_0)$.

Example 1.10. When ν is defined by a formal curve V, then $\Pi[\nu]$ is the sequence of infinitely nearby points associated to V.

Example 1.11. If $\nu = \nu_{E,p}$ is an exceptional curve valuation, then $\Pi[\nu] = (p_i)_0^\infty$, where $\Pi[\nu_E] = (p_i)_0^{k-1}$, $p_k = p$ and, inductively, p_{j+1} is the intersection point of E_{j+1} and the strict transform of $E = E_k$.

Example 1.12. Let $s \geq 1$, and $\nu = \nu_s$ the monomial valuation with $\nu(x) = 1$, $\nu(y) = s$. Then $(p_i) = \Pi[\nu]$ is determined by the continued fraction expansion

$$s = a_1 + \frac{1}{a_2 + \dots} = [[a_1, a_2, \dots]].$$

See [Sp] for details.

Part 2. MacLane's method

In this part we do the preparatory work for obtaining the tree structure on valuation space. Namely, we show how to represent a valuation conveniently by a finite or countable sequence of polynomials and real numbers.

Our approach is an adaptation of MacLane's method [M] and can be outlined as follows. Pick a centered valuation ν on $\mathbf{C}[[x,y]]$. It corresponds uniquely to a valuation, still denoted ν, on the Euclidean domain $\mathbf{C}(x)[y]$. Write $U_0 = x$ and $U_1 = y$. The value $\tilde\beta_0 = \nu(U_0)$ determines ν on $\mathbf{C}(x)$ and the value $\tilde\beta_1 = \nu(U_1)$ determines ν on many polynomials in y. The idea is now to find a polynomial $U_2(x,y)$ of minimal degree in y such that $\tilde\beta_2 = \nu(U_2)$ is not determined by the previous data. Inductively we construct polynomials U_j and numbers $\tilde\beta_j = \nu(U_j)$ which, as it turns out, represent ν completely.

More precisely we proceed as follows. First we define what data $[(U_j);(\tilde\beta_j)]$ to use, namely sequences of toroidal key polynomials, or STKP's. These are introduced in Section 2 where we also show how to associate a canonical valuation to an STKP. In Section 3 we study such a valuation in detail, describing its graded ring and computing its numerical invariants. Then in Section 4 we show that in fact any valuation is associated to an STKP. Finally, in Section 5 we compute $\nu(\phi)$ for $\phi \in \mathfrak{m}$ irreducible in terms of the STKP's associated to ν and ν_ϕ. The latter computation is the key to the metric tree structure of valuation space as described in Section 8.

We formulate almost all results for valuations rather than Krull valuations, but the method works for both classes. We leave it to the reader to make the suitable adaptation in the notation and statements.

2. Sequences of toroidal key polynomials.

2.1. Key polynomials.

Let us define the data we use to represent valuations.

Definition 2.1. A sequence of polynomials $(U_j)_{j=0}^k$, $1 \le k \le \infty$, in $\mathbf{C}[x,y]$ is called a *sequence of toroidal key-polynomials* (STKP) if it satisfies:

(P0) $U_0 = x$ and $U_1 = y$;

(P1) to each polynomial U_j is attached a number $\tilde\beta_j \in \overline{\mathbf{R}}_+$ (not all ∞) with

$$\tilde\beta_{j+1} > n_j\tilde\beta_j = \sum_{l=0}^{j-1} m_{j,l}\tilde\beta_l \quad \text{for } 1 \le j < k, \tag{2.1}$$

where $n_j \in \mathbf{N}^*$ and $m_{j,i} \in \mathbf{N}$ satisfy, for $j > 1$ and $1 \le i < j$,

$$n_j = \min\{l \in \mathbf{Z} \mid l\,\tilde\beta_j \in \mathbf{Z}\tilde\beta_0 + \cdots + \mathbf{Z}\tilde\beta_{j-1}\} \quad \text{and} \quad 0 \le m_{j,i} < n_i; \tag{2.2}$$

(P2) for $1 \le j < k$ there exists $\theta_j \in \mathbf{C}^*$ such that

$$U_{j+1} = U_j^{n_j} - \theta_j \cdot U_0^{m_{j,0}} \cdots U_{j-1}^{m_{j,j-1}} \tag{2.3}$$

In the sequel, we call k the *length* of the STKP.

Remark 2.2. If an abstract semi-group Γ is given, a sequence of polynomials satisfying (P0)-(P2) with $\tilde\beta_j \in \Gamma$ will be called a Γ-STKP.

Remark 2.3. If $(U_j)_0^k$ is an STKP of length $k \geq 2$, then $(\tilde{\beta}_j)_1^{k-1}$ are determined by $\tilde{\beta}_0$ and $(U_j)_0^k$. Conversely, given a sequence $(\tilde{\beta}_j)_1^{k+1}$ satisfying (P2) and any sequence $(\theta_j)_1^k$ in $\mathbf{C}^*$ there exists a unique associated STKP. Despite this redundancy we will typically write an STKP as $[(U_j); (\tilde{\beta}_j)]$.

Lemma 2.4. *For $1 \leq j \leq k$, the polynomial U_j is irreducible and of Weierstrass form $U_j = y^{d_j} + a_1(x)y^{d_j-1} + \cdots + a_{d_j}(x)$ with $a_l(0) = 0$ for all l. Moreover, $d_{j+1} = n_j d_j$ for $1 \leq j < k$.*

Proof. The proof of all assertions is by induction. The fact that U_j is irreducible is not obvious and will follow from the proof of Theorem 2.7. We included this fact here for clarity. Let us show by induction on k that $d_{j+1} = n_j d_j$ for $1 \leq j < k$.

By (2.2) and the induction hypothesis, we have $m_{j,i} < n_i = d_{i+1}/d_i$, hence $m_{j,i} + 1 \leq d_{i+1}/d_i$ as both sides are integers. We infer

$$\sum_{i=1}^{j-1} m_{j,i} d_i \leq \sum_{i=1}^{j-1} (d_{i+1}/d_i - 1) d_i = d_j - 1 < n_j d_j,$$

hence $\deg_y(U_{j+1}) = n_j d_j$. □

Remark 2.5. In the notation of Zariski and Spivakovsky the sequence $(\overline{\beta}_i)$ can be extracted from $(\tilde{\beta}_i)$ as follows. One has $\overline{\beta}_i = \tilde{\beta}_{r_i}$ where r_i is defined inductively to be the smallest integer $r_i > r_{i-1}$ so that $n_{r_i} \geq 2$. In the Abhyankar-Moh terminology [AM], the U_{r_i} are the approximate roots of U_k.

We include a result of arithmetic nature which will be used repeatedly.

Lemma 2.6. *Assume $\tilde{\beta}_0, \ldots, \tilde{\beta}_{k+1}$ are given so that for all $j = 1, \ldots k$ one has $\tilde{\beta}_{j+1} > n_j \cdot \tilde{\beta}_j$ with $n_j := \min\{l \in \mathbf{N}^* \mid l\tilde{\beta}_j \in \sum_0^{j-1} \mathbf{Z}\tilde{\beta}_i\}$. Then for any $j = 1, \ldots, k$ there exists a unique decomposition $n_j\tilde{\beta}_j = \sum_0^{j-1} m_{j,l}\tilde{\beta}_l$ where $m_{j,l}$ are* non-negative *integers, and $m_{j,l} < n_l$ for $l = 1, \ldots, j-1$.*

Proof. By assumption there exist $m_i \in \mathbf{Z}$ with $n_j\tilde{\beta}_j = \sum_0^{j-1} m_l\tilde{\beta}_l$. By Euclidean division, $m_{j-1} = q_{j-1}n_{j-1} + r_{j-1}$ with $0 \leq r_{j-1} < n_{j-1}$ so by the fact that $n_{j-1} \cdot \tilde{\beta}_{j-1} \in \sum_0^{j-2} \mathbf{Z} \cdot \tilde{\beta}_i$ we can suppose that $0 \leq m_{j-1} < n_{j-1}$. Inductively we get $0 \leq m_i < n_i$ for $1 \leq i < j$. It remains to prove that $m_0 \geq 0$. Set $S = \sum_1^{j-1} m_l\tilde{\beta}_l$. We have $m_1\tilde{\beta}_1 < n_1\tilde{\beta}_1 < \tilde{\beta}_2$ and $m_2 < n_2$ so $m_2 + 1 \leq n_2$ and

$$m_1\tilde{\beta}_1 + m_2\tilde{\beta}_2 < (1 + m_2)\tilde{\beta}_2 \leq n_2\tilde{\beta}_2 < \tilde{\beta}_3.$$

We infer that $S < (1 + m_{j-1})\tilde{\beta}_{j-1} \leq n_{j-1}\tilde{\beta}_{j-1} < \tilde{\beta}_j$ so that $m_0\tilde{\beta}_0 \geq \tilde{\beta}_j - S \geq 0$. □

2.2. **From STKP's to valuations I.** We show how to associate a valuation ν to any finite STKP in a canonical way.

Theorem 2.7. *Let $[(U_j)_0^k; (\tilde{\beta}_j)_j^k]$ be an STKP of length $1 \leq k < \infty$. Then there exists a unique centered valuation $\nu_k \in \tilde{V}$ satisfying*

(Q1) *$\nu_k(U_j) = \tilde{\beta}_j$ for $0 \leq j \leq k$;*

(Q2) *$\nu_k \leq \nu$ for any $\nu \in \tilde{V}$ satisfying (Q1).*

Further, if $k > 1$ and ν_{k-1} is the valuation associated to $[(U_j)_0^{k-1}; (\tilde{\beta}_j)_0^{k-1}]$, then

(Q3) $\nu_{k-1} \leq \nu_k$;

(Q4) $\nu_{k-1}(\phi) < \nu_k(\phi)$ *if and only if* U_k *divides* ϕ *in* $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.

Remark 2.8. The valuation ν_k is normalized, i.e. $\nu_k \in \mathcal{V}$, iff $\min\{\tilde{\beta}_0, \tilde{\beta}_1\} = 1$.

The proof of Theorem 2.7 occupies all of Section 2.3. It is a subtle induction on k using divisibility properties in the graded ring $\mathrm{gr}_\nu\mathbf{C}(x)[y]$.

First, ν_1 is defined to be the monomial valuation with $\nu_1(x) = \tilde{\beta}_0, \nu_1(y) = \tilde{\beta}_1$, i.e. $\nu_1(\sum a_{ij}x^iy^j) = \min\{i\tilde{\beta}_0 + j\tilde{\beta}_1 \mid a_{ij} \neq 0\}$. Property (Q2) clearly holds.

Now assume $k > 1$, that the STKP $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ is given and that $\nu_1, \dots, \nu_{k-1}$ have been defined. First consider a polynomial $\phi \in \mathbf{C}[x,y]$. As U_k is unitary in y, we can divide ϕ by U_k in $\mathbf{C}[x][y]$: $\phi = \phi_0 + U_k\psi$ with $\deg_y(\phi_0) < d_k = \deg_y(U_k)$ and $\psi \in \mathbf{C}[x,y]$. Iterating the procedure we get a unique decomposition

$$\phi = \sum_j \phi_j U_k^j \tag{2.4}$$

with $\phi_j \in \mathbf{C}[x,y]$ and $\deg_y(\phi_j) < d_k$. Define

$$\nu_k(\phi) := \min_j \nu_k(\phi_j U_k^j) := \min_j\{\nu_{k-1}(\phi_j) + j\tilde{\beta}_k\}. \tag{2.5}$$

The function ν_k is easily seen to satisfy (V2) and (V3). We will show it also satisfies (V1), hence defines a valuation on $\mathbf{C}[x,y]$, which automatically satisfies (Q2). Theorem 2.7 then follows from

Proposition 2.9 ([Sp])**.** *Any centered valuation $\nu : \mathbf{C}[x,y] \to \overline{\mathbf{R}}_+$ has a unique extension to a valuation on $R = \mathbf{C}[[x,y]]$. This extension preserves the partial ordering: if $\nu_1(\phi) \leq \nu_2(\phi)$ holds for polynomials ϕ, it also holds for formal power series.*

Proof. We may assume $\nu \in \mathcal{V}$ is normalized, $\nu(\mathfrak{m}) = 1$. Pick $\phi \in R$, and write $\phi = \phi_k + \widehat{\phi}_k$ where ϕ_k is the truncation of ϕ at order k. Then $\nu(\phi) := \lim_k \nu(\phi_k)$ exists in $\overline{\mathbf{R}}_+$. Indeed, for any k, l, we have

$$\nu(\phi_{k+l}) \geq \min\{\nu(\phi_k), \nu(\phi_{k+l} - \phi_k)\} \geq \min\{\nu(\phi_k), k\}$$

Thus the sequence $\min\{\nu(\phi_k), k\}$ is nondecreasing, hence converges.

Uniqueness can be proved as follows: $\nu(\phi) \geq \min\{\nu(\phi_m), m\}$ for all m. If $\nu(\phi_m)$ is unbounded, then $\nu(\phi) = \infty$. Otherwise $\nu(\phi) = \nu(\phi_m)$ for m large enough. By construction the extension preserves the ordering of valuations. □

The extension above need not preserve numerical invariants. For example, if ν is a curve valuation at a non-algebraic curve, then $\mathrm{rk}(\nu|_{\mathbf{C}[x,y]}) = 1$ but $\mathrm{rk}(\nu) = 2$.

Remark 2.10. The expansion (2.4) holds for any $\phi \in \mathbf{C}[[x,y]]$. Using Weierstrass' preparation theorem, we can assume that ϕ is a polynomial in y. Weierstrass' division theorem then yields (2.4) with $\phi_j \in \mathbf{C}[[x]][y]$. The valuation ν_k can be hence defined by (2.5) for any formal power series.

2.3. **Proof of Theorem 2.7.** By Proposition 2.9, we will restrict our attention to centered valuations defined on the Euclidean ring $\mathbf{C}(x)[y]$. We first make the induction hypothesis precise.

(H_k): ν_k is a valuation satisfying (Q1)-(Q4).
(E_k): the graded ring $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ is a Euclidean domain.
(I_k): U_k and U_{k+1} are irreducible in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.
($\overline{I}_k$): U_j is irreducible both in $\mathbf{C}[x,y]$, and $\mathbf{C}(x)[y]$ for $1 \le j \le k$.

For the precise definition of a Euclidean domain we refer to [ZS] p.23. One immediately checks that H_1 and $\overline{I}_1$ hold. Our strategy is to show successively $\overline{I}_k$ & $H_k \Rightarrow E_k$; $\overline{I}_k$ & $H_k \Rightarrow I_k$; E_k & I_k & $H_k \Rightarrow H_{k+1}$; I_k & $\overline{I}_k$ & $H_{k+1} \Rightarrow \overline{I}_{k+1}$.

Step 1: $\overline{I}_k$ & $H_k \Rightarrow E_k$. We want to show that $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ is a Euclidean domain. Let $\phi \in R$ and consider the expansion (2.4). We define

$$\delta_k(\phi) := \max\left\{ j \mid \nu_k(\phi_j U_k^j) = \nu_k(\phi) \right\}.$$

By convention we let $\delta_k(0) = -\infty$. Note that if $\phi = \phi'$ modulo ν_k, then $\delta_k(\phi) = \delta_k(\phi')$. Hence δ_k is well defined on $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.

Lemma 2.11. *For all $\phi, \psi \in \mathbf{C}(x)[y]$, $\delta_k(\phi\psi) = \delta_k(\phi) + \delta_k(\psi)$.*

Proof. First assume $\delta_k(\phi) = \delta_k(\psi) = 0$. Then $\phi = \phi_0$, $\psi = \psi_0$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$, so we can assume $\deg_y(\phi), \deg_y(\psi) < d_k$. Thus $\deg_y(\phi\psi) < 2d_k$ so $\phi\psi = \alpha_0 + \alpha_1 U_k$ with $\deg_y(\alpha_0), \deg_y(\alpha_1) < d_k$. Suppose $\nu_k(\alpha_1 U_k) = \nu_k(\phi\psi)$. Then $\nu_{k-1}(\alpha_1 U_k) < \nu_k(\alpha_1 U_k) \le \nu_k(\alpha_0) = \nu_{k-1}(\alpha_0)$. Thus

$$\nu_{k-1}(\alpha_1 U_k) = \nu_{k-1}(\phi\psi) = \nu_{k-1}(\phi) + \nu_{k-1}(\psi) = \nu_k(\phi) + \nu_k(\psi) = \nu_k(\phi\psi).$$

Hence $\nu_{k-1}(\alpha_1 U_k) = \nu_k(\alpha_1 U_k)$ contradicting (Q4). So $\phi\psi = \alpha_0$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ and $\delta_k(\phi\psi) = 0$.

In the general case, $\phi = \sum \phi_i U_k^i$, $\psi = \sum \psi_j U_k^j$, we have $\phi\psi = \sum(\phi_i\psi_j)U_k^{i+j}$. By what precedes $\delta_k(\phi_i\psi_j) = 0$ for all i, j so $\delta_k(\phi\psi) = \delta_k(\phi) + \delta_k(\psi)$. $\square$

Lemma 2.12. *For $\phi \in \mathbf{C}(x)[y]$, $\delta_k(\phi) = 0$ iff ϕ is a unit in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.*

Proof. If $\delta_k(\phi) = 0$, then $\phi = \phi_0$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$. As U_k is irreducible in $\mathbf{C}(x)[y]$ and $\deg_y(U_k) > \deg_y(\phi_0)$, the polynomial U_k is prime with ϕ_0. Hence we can find $A, B \in \mathbf{C}(x)[y]$ with $\deg_y A, \deg_y B < d_k$ so that $A\phi_0 = 1 - BU_k$. Then, comparing with (2.4), we have $\nu_k(A\phi_0) = \nu_k(1) < \nu_k(BU_k)$. Therefore $A\phi_0 = 1$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ so ϕ_0, and hence ϕ, is a unit in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.

Conversely, if ϕ is a unit, say $A\phi = 1$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ for some $A \in \mathbf{C}(x)[y]$, then $\delta_k(\phi) + \delta_k(A) = \delta_k(1) = 0$, so $\delta_k(\phi) = 0$. $\square$

Lemma 2.13. *If $\phi, \psi \in \mathbf{C}(x)[y]$, then there exists $Q, R \in \mathbf{C}(x)[y]$ such that $\phi = Q\psi + R$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ and $\delta_k(R) < \delta_k(\psi)$.*

Proof. Write $\psi = \sum_j \psi_j U_k^j$. It suffices to prove the lemma when $\psi_j = 0$ for $j > M := \delta_k(\psi)$ and using Lemma 2.12 we may assume $\psi_M = 1$. As $\deg_y(\psi_j) < d_k$ for $j \le M$ we have $\deg_y(\psi) = Md_k$. Euclidean division in $\mathbf{C}(x)[y]$ yields $Q, R^1 \in$

$\mathbf{C}(x)[y]$ with $\deg_y(R^1) < \deg_y(\psi)$ so that $\phi = Q\psi + R^1$. Write $R^1 = \sum_i R_i U_k^i$ and set $N := \delta_k(R^1)$, $R := \sum_{i\le N} R_i U_k^i$. Then $\phi = Q\psi + R$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ and

$$\deg_y(R) = \deg_y(R_N) + Nd_k < Md_k = \deg_y(\psi).$$

Hence $N < M$ and we are done. □

This completes Step 1. The Euclidean property of $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ makes every ideal principal and supplies us with unique factorization and Gauss' lemma.

Step 2: $\overline{I}_k$ & $H_k \Rightarrow I_k$. We want to show that U_k and U_{k+1} are irreducible in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ and proceed in several steps.

Lemma 2.14. *U_k is irreducible in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.*

Proof. We trivially have $\delta_k(U_k) = 1$ so if $\phi\psi = U_k$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$, then $\delta_k(\phi) = 0$ or $\delta_k(\psi) = 0$. Hence ϕ or ψ is a unit in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ (see Lemma 2.11 and 2.12). □

Lemma 2.15. *If $j < k$ then U_j is a unit in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.*

Proof. By Lemma 2.12 it suffices to show that $\delta_k(U_j) = 0$. If $d_j < d_k$ then this is obvious. If $d_j = d_k$, then $U_j = (U_j - U_k) + U_k$, where $\deg_y(U_j - U_k) < d_k$. Now $\nu_k(U_j) = \tilde\beta_j < \tilde\beta_k = \nu_k(U_k)$, so $\nu_k(U_j - U_k) < \nu_k(U_k)$ and $\delta_k(U_j) = 0$. □

Lemma 2.16. *If $\delta_k(\phi) < n_k$, then $\phi = \phi_i U_k^i$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ for some $i < n_k$.*

Proof. Suppose $\nu_k(\phi_i U_k^i) = \nu_k(\phi_j U_k^j) = \nu_k(\phi)$, where $i \le j < n_k$. Then $(j - i)\tilde\beta_k = \nu_{k-1}(\phi_i) - \nu_{k-1}(\phi_j) \in \sum_0^{k-1} \mathbf{Z}\tilde\beta_j$. By (2.2) n_k divides $j - i$ hence $i = j$. □

Lemma 2.17. *U_{k+1} is irreducible in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.*

Proof. We have $U_{k+1} = U_k^{n_k} - \tilde U_{k+1}$, where $\tilde U_{k+1} = \theta_k \prod_{j=0}^{k-1} U_j^{m_{k,j}}$. Assume $U_{k+1} = \phi\psi$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ with $0 < \delta_k(\phi), \delta_k(\psi) < n_k$. By Lemma 2.16, we can write $\phi = \phi_i U_k^i$, $\psi = \psi_j U_k^j$. Then $U_{k+1} = \phi_i\psi_j U_k^{n_k}$ so $(1 - \phi_i\psi_j)U_k^{n_k} = \tilde U_{k+1}$. As U_k is irreducible and $\tilde U_{k+1}$ is a unit, we have $\phi_i\psi_j = 1$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$. But then $\tilde U_{k+1} = 0$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$, which is absurd. So we can assume $\delta_k(\phi) = n_k$ and $\delta_k(\psi) = 0$. Hence ψ is a unit, completing the proof. □

Step 3: E_k & I_k & $H_k \Rightarrow H_{k+1}$. We must show $\nu_{k+1}(\phi\psi) = \nu_{k+1}(\phi) + \nu_{k+1}(\psi)$ for $\phi, \psi \in \mathbf{C}[x, y]$ where ν_{k+1} is defined as in (2.5). We first note

(i) $\deg_y(\phi) < d_{k+1}$ implies $\nu_{k+1}(\phi) = \nu_k(\phi)$;
(ii) $\nu_k(U_{k+1}) = n_k\tilde\beta_k < \tilde\beta_{k+1} = \nu_{k+1}(U_{k+1})$;
(iii) $\nu_{k+1}(U_{k+1}\phi) = \nu_{k+1}(U_{k+1}) + \nu_{k+1}(\phi)$ for all $\phi \in \mathbf{C}[x, y]$.

The second assertion follows from (2.1). We now show

Lemma 2.18. *For $\phi \in \mathbf{C}[x, y]$ we have $\nu_{k+1}(\phi) \ge \nu_k(\phi)$.*

Proof. We argue by induction on $\deg_y(\phi)$. If $\deg_y(\phi) < d_{k+1}$, then we are done by (i). If $\deg_y(\phi) \ge d_{k+1}$, then we write $\phi = \sum_i \phi_i U_{k+1}^i$ as in (2.4). By the induction hypothesis and (ii)-(iii) above we may assume that $\phi_0 \not\equiv 0$. Write $\phi = \phi_0 + \psi$ with $\psi = \sum_{i\ge1} \phi_i U_{k+1}^i$. When $\nu_k(\phi) = \min\{\nu_k(\phi_0), \nu_k(\psi)\}$, one has

$$\nu_{k+1}(\phi) = \min\{\nu_{k+1}(\phi_0), \nu_{k+1}(\psi)\} \ge \nu_k(\phi),$$

proving the lemma in this case. Otherwise, $\phi_0+\psi = 0$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$. This implies that U_{k+1} divides ϕ_0 in this ring. But $\deg_y(\phi_0) < d_{k+1}$, hence $\delta_k(\phi_0) < n_k$, so that ϕ_0 is a power of the irreducible U_k times a unit in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ by Lemma 2.16. By Lemma 2.17, U_{k+1} is also irreducible, a contradiction. □

Introduce $\mathfrak{p} := \{\nu_{k+1} > \nu_k\} \subset \mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$. Using the preceding lemma, one easily verifies that $\mathfrak{p}$ is a proper ideal, which contains the irreducible element U_{k+1} by (ii). As $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ is a Euclidean domain, $\mathfrak{p}$ is generated by U_{k+1}.

Fix $\phi, \psi \in R$. We want to show that $\nu_{k+1}(\phi\psi) = \nu_{k+1}(\phi) + \nu_{k+1}(\psi)$. First assume $\phi, \psi \notin \mathfrak{p}$. Then $\phi\psi \notin \mathfrak{p}$. By H_k, ν_k is a valuation, so

$$\nu_{k+1}(\phi\psi) = \nu_k(\phi\psi) = \nu_k(\phi) + \nu_k(\psi) = \nu_{k+1}(\phi) + \nu_{k+1}(\psi).$$

In the general case, write $\phi = \widehat{\phi}\, U_{k+1}^n$, $\psi = \widehat{\psi}\, U_{k+1}^m$ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ where $\widehat{\phi}$, $\widehat{\psi}$ are prime with U_{k+1}. In particular they do not belong to $\mathfrak{p}$ so that

$$\begin{aligned}\nu_{k+1}(\phi\psi) &= \nu_{k+1}(\widehat{\phi}\,\widehat{\psi}\, U_{k+1}^{m+n}) \\ &= \nu_{k+1}(\widehat{\phi}) + \nu_{k+1}(\widehat{\psi}) + (m+n)\nu_{k+1}(U_{k+1}) = \nu_{k+1}(\phi) + \nu_{k+1}(\psi).\end{aligned}$$

This completes Step 3.

Step 4: I_k & $\overline{I}_k$ & $H_{k+1} \Rightarrow \overline{I}_{k+1}$. What we have to prove is

Lemma 2.19. *U_{k+1} is irreducible in $\mathbf{C}[x,y]$ and $\mathbf{C}(x)[y]$.*

Proof. As U_{k+1} is monic in y, irreducibility in $\mathbf{C}[x,y]$ implies irreducibility in $\mathbf{C}(x)[y]$. So suppose $\phi\phi' = U_{k+1}$ for $\phi, \phi' \in \mathbf{C}[x,y]$. As U_{k+1} is irreducible in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$, we may assume that the image of ϕ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ is a unit i.e. $\delta_k(\phi) = 0$ and $\delta_k(\phi') = n_k$. On the other hand, one has $\delta_k(\psi) d_k \leq \deg_y(\psi)$ for any $\psi \in \mathbf{C}[x,y]$. Using Lemma 2.4 (except the assertion of irreducibility which we want now to prove), we infer

$$d_k\delta_k(\phi') = d_k n_k = \deg_y(U_{k+1}) = \deg_y(\phi) + \deg_y(\phi') \geq \deg_y(\phi') \geq d_k\delta_k(\phi'),$$

hence $\deg_y(\phi) = 0$ and $\deg_y(\phi') = \deg_y(U_{k+1})$. Now U_{k+1} is unitary in y so $\phi \notin \mathfrak{m}$ and is hence a unit in R. □

This completes the proof of Theorem 2.7.

Remark 2.20. Since U_{k+1} is monic in y we may use Weierstrass' division theorem and the argument above to show that U_{k+1} is also irreducible in $\mathbf{C}[[x,y]]$.

2.4. **From STKP's to valuations II.** We now turn to infinite STKP's.

Theorem 2.21. *Let $[(U_j);(\tilde{\beta}_j)]$ be an infinite STKP and let ν_k the valuation associated to $[(U_j)_0^k;(\tilde{\beta}_j)_0^k]$ for $k \geq 1$ by Theorem 2.7.*

(i) *If $n_j \geq 2$ for infinitely many j, then for any $\phi \in R$ there exists $k_0 = k_0(\phi)$ such that $\nu_k(\phi) = \nu_{k_0}(\phi)$ for all $k \geq k_0$. In particular, ν_k converges to a valuation ν_∞.*

(ii) *If $n_j = 1$ for $j \gg 1$, then U_k converges in $\mathbf{C}[[x,y]]$ to an irreducible formal power series U_∞ and ν_k converges to a valuation ν_∞. More precisely, for $\phi \in R$ prime to U_∞ we have $\nu_k(\phi) = \nu_{k_0}(\phi)$ for $k \geq k_0 = k_0(\phi)$, and if U_∞ divides ϕ, then $\nu_k(\phi) \to \infty$.*

Proof of Theorem 2.21. If $n_j \geq 2$ for infinitely many j, then $\deg_y(U_{k+1}) = d_{k+1} = \prod_1^k n_j$ tends to infinity. Pick $\phi \in R$. By Weierstrass' division theorem we may assume $\deg_y(\phi) < \infty$. By Remark 2.10, $\nu_k(\phi)$ is defined using formula (2.4). For $k \gg 1$, $\deg_y(\phi) < \deg_y(U_{k+1})$, so that $\nu(\phi) = \nu(\phi) = \nu_k(\phi) = \nu_{k+l}(\phi) = \nu_{k+l}(\phi)$ for all $l \geq 0$. Thus ν_k converges towards a valuation ν_∞.

If $n_k = 1$ for $k \geq K$, then set $d := \max_j \deg_y(U_j)$. For $k \geq K$,

$$U_{k+1} = U_k - \theta_k \prod_0^{k-1} U_j^{m_{k,j}}. \tag{2.6}$$

As $\deg_y(U_{k+1}) = \deg_y(U_k) = d$ one has $m_{k,j} = 0$ for any $j \geq K$ by (2.2). Then $\{\tilde{\beta}_k\}_{k \geq K}$ is a strictly increasing sequence of real numbers belonging to the discrete lattice $\sum_0^K \mathbf{Z}\tilde{\beta}_j$, so $\tilde{\beta}_k \to \infty$. Write

$$U_k = y^d + a_{d-1}^k(x) y^{d-1} + \cdots + a_0^k(x),$$

the Weierstrass form of U_k. As $m_{k,j} = 0$ for any $j \geq K$ we get

$$a_n^{k+1}(x) = a_n^k(x) - \theta_k \sum_I a_{i_1}^{j_1}(x) \ldots a_{i_l}^{j_l}(x)$$

where $\#\{j = \alpha \mid j \in I\} = m_{k,\alpha}$, and $i_1 + \cdots + i_l = n$. Hence $a_n^{k+1}(x) - a_n^k(x) \in \mathfrak{m}^{\sum_0^K m_{k,i}}$. But the sequence $\tilde{\beta}_j$ is increasing so

$$\sum_0^K m_{k,i} \geq \tilde{\beta}_K^{-1} \sum_0^K \tilde{\beta}_i m_{k,i} = \frac{\tilde{\beta}_k}{\tilde{\beta}_K} \to \infty,$$

and U_k converges towards a polynomial in y that we denote by U_∞. By (2.6) we have $U_{k+1} = U_k$ modulo ν_{k-1} and $U_{k+l} = U_k$ modulo ν_{k-1} for all $l \geq 0$. So $U_\infty = U_k$ in ν_{k-1}. Therefore $\nu_\infty(U_\infty) := \lim \nu_k(U_\infty) = \lim \nu_k(U_k) = \lim \tilde{\beta}_k = \infty$. Let $\phi \in R$ and write $\phi = \phi_0 + U_\infty \phi_1$ with $\deg_y \phi_0 < \deg_y U_\infty = d$. When $\phi_0 \equiv 0$, then set $\nu_\infty(\phi) := \lim \nu_k(\phi) = \infty$. Otherwise for $k \geq K$ large enough, for all $l \geq 0$, $\nu_{k+l}(\phi) = \nu_{k+l}(\phi_0) = \nu_k(\phi_0)$. Hence the sequence $\nu_k(\phi)$ is stationary for $k \geq K$ and we can set $\nu_\infty(\phi) := \lim \nu_k(\phi)$. One easily checks that ν_∞ is a valuation. As $\nu_\infty(\phi) = \infty$ iff $U_\infty | \phi$ it follows that U_∞ is irreducible in $\mathbf{C}[[x, y]]$. □

3. Graded rings and numerical invariants

The aim of this section is to give the structure of the valuation associated to a finite or infinite STKP. That is, we describe the structure of the graded ring $\mathrm{gr}_\nu \mathbf{C}(x)[y]$, and compute the three invariants $\mathrm{rk}(\nu)$, $\mathrm{rat.rk}(\nu)$ and $\mathrm{tr.deg}(\nu)$. We refer to [T2] for general results in higher dimensions.

Given a finite or infinite STKP we denote by $\nu := \mathrm{val}\,[(U_j); (\tilde{\beta}_j)]$ its associated valuation through Theorem 2.7 or 2.21.

3.1. **Homogeneous decomposition I.**
The following theorem gives the structure of the graded ring of a valuation defined by a finite STKP.

Theorem 3.1. *Let* $\nu := \mathrm{val}\,[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$, *where* $1 \leq k < \infty$, *define* $n_k := \min\{j > 0 \mid j \cdot \tilde{\beta}_k \in \sum_0^{k-1} \mathbf{Z} \cdot \tilde{\beta}_j\}$, *and pick* $0 \neq \phi \in R = \mathbf{C}[[x, y]]$.

(i) *If* $n_k = \infty$ *or equivalently* $\tilde\beta_k \notin \mathbf{Q}\cdot\tilde\beta_0$, *then*

$$\phi = \alpha \prod_{j=0}^{k} U_j^{i_j} \text{ in } \mathrm{gr}_\nu R \text{ and } \mathrm{gr}_\nu R_\nu \tag{3.1}$$

with $\alpha \in \mathbf{C}^*$, $0 \le i_j < n_j$ *for* $1 \le j < k$ *and* $i_0, i_k \ge 0$.

(ii) *If* $n_k < \infty$, *write* $n_k\tilde\beta_k = \sum_0^{k-1} m_{k,j}\tilde\beta_j$ *with* $0 \le m_{k,j} < n_j$ *for* $1 \le j < k$ *and* $m_0 \ge 0$ *as in Lemma 2.6. Then*

$$\phi = p(T) \prod_{j=0}^{k} U_j^{i_j} \text{ in } \mathrm{gr}_\nu R_\nu \tag{3.2}$$

with $0 \le i_j < n_j$ *for* $1 \le j \le k$, $i_0, i_k \ge 0$ *and where* p *is a polynomial in* $T := U_k^{n_k}\prod_{j=0}^{k-1} U_j^{-m_{k,j}}$.

Both decompositions (3.1), (3.2) *are unique.*

Recall that the ring $\mathrm{gr}_\nu\mathbf{C}(x)[y]$ is a Euclidean domain. Let us describe its irreducible elements.

Corollary 3.2. *Let* ν *be a valuation as above.*

(i) *When* $n_k = \infty$ *the only irreducible element of* $\mathrm{gr}_\nu\mathbf{C}(x)[y]$ *is* U_k.

(ii) *When* $n_k < \infty$ *the irreducible elements of* $\mathrm{gr}_\nu\mathbf{C}(x)[y]$ *consist of* U_k *and all elements of the form* $U_k^{n_k} - \theta\prod_{j=0}^{k-1} U_j^{m_{k,j}}$ *for some* $\theta \in \mathbf{C}^*$.

Proof of Corollary 3.2. Assume $\phi \in \mathrm{gr}_\nu\mathbf{C}(x)[y]$ is irreducible. If $n_k = \infty$, then $\phi = \alpha\prod_{j=0}^{k} U_j^{i_j}$ by (3.1). But U_j is a unit for $j < k$ by Lemma 2.15, so U_k is the only irreducible element in $\mathrm{gr}_\nu\mathbf{C}(x)[y]$.

When $n_k < \infty$, then we use (3.2) and factorize $p(T) = \prod(T - \theta_l)$. Modulo unit factors, we hence get

$$\phi = U_k^{i_k - Ln_k}\prod_l \left(U_k^{n_k} - \theta_l \prod_j U_j^{m_{k,j}}\right), \tag{3.3}$$

where $L = \deg p$. On the other hand Lemma 2.17 shows that all elements of the form $U_k^{n_k} - \theta_l\prod_j U_j^{m_{k,j}}$ are irreducible in $\mathrm{gr}_\nu\mathbf{C}(x)[y]$. So (3.3) is the decomposition of ϕ into prime factors in $\mathrm{gr}_\nu\mathbf{C}(x)[y]$. This concludes the proof. □

Proof of Theorem 3.1. First assume that $\deg_y\phi < d_k = \deg_y U_k$. We will show that (3.1) holds. Write $\phi = \sum_0^{n_{k-1}-1}\phi_i U_{k-1}^i$ with $\deg_y\phi_i < d_{k-1}$ for all i. Iterating this procedure we get $\phi = \sum\alpha_I U_0^{i_0}\cdots U_{k-1}^{i_{k-1}}$ with $\alpha_I \in \mathbf{C}$, $i_0 \ge 0$ and $0 \le i_j < n_j$ for $j \ge 1$. If $\nu(U_0^{i_0}\dots U_{k-1}^{i_{k-1}}) = \nu(U_0^{j_0}\dots U_{k-1}^{j_{k-1}})$ then $\sum i_l\tilde\beta_l = \sum j_l\tilde\beta_l$ so that $(i_{k-1} - j_{k-1})\tilde\beta_{k-1} \in \sum_0^{k-2}\mathbf{Z}\tilde\beta_j$. Thus n_{k-1} divides $i_{k-1} - j_{k-1}$ so $i_{k-1} = j_{k-1}$ as $0 \le i_{k-1}, j_{k-1} < n_{k-1}$. Iterating this argument we get (3.1).

In the general case, we can assume $\deg_y\phi < \infty$ by Weierstrass preparation theorem. Write $\phi = \sum\phi_i U_k^i$, with $\deg_y\phi_i < d_k$. By what precedes, we have

$$\phi = \sum\alpha_I U_0^{i_0}\dots U_{k-1}^{i_{k-1}}U_k^{i_k} \text{ in } \mathrm{gr}_\nu R$$

with $\alpha_I \in \mathbf{C}^*$, $0 \le i_j < n_j$ for $1 \le j < k$, $i_0, i_k \ge 0$. We may assume $\sum_0^k i_j \tilde\beta_j = \nu(\phi)$ for all I with $\alpha_I \ne 0$. If (3.1) is not valid, there exist $I \ne J$ with $\alpha_I, \alpha_J \ne 0$. Thus $(i_k - j_k)\tilde\beta_k \in \sum_0^{k-1} \mathbf{Z}\tilde\beta_j$ implying $n_k < \infty$. This shows (3.1) when $n_k = \infty$.

Now assume $n_k < \infty$, and write $n_k\tilde\beta_k = \sum_0^{k-1} m_{k,j}\tilde\beta_j$ as in Lemma 2.6. Let $I = (i_0, \dots, i_k)$ be any multiindex with $\alpha_I \ne 0$. Make the Euclidean division $i_k = r_k n_k + \widehat{\imath}_k$ with $0 \le \widehat{\imath}_k < n_k$, and write

$$\prod_{j=0}^{k} U_j^{i_j} = U_k^{\widehat{\imath}_k} T^{r_k} \prod_{j=0}^{k-1} U_j^{a_j}$$

with $a_j := i_j + r_k m_{k,j}$ and $T := U_k^{n_k} \prod_{j=0}^{k-1} U_j^{-m_{k,j}}$. The key remark is now that $U_j^{n_j} = \theta_j \prod_0^{j-1} U_l^{m_{j,l}}$ in $\mathrm{gr}_\nu R$ for $1 \le j < k$. Making the Euclidean division $a_{k-1} = r_{k-1}n_{k-1} + \widehat{\imath}_{k-1}$ with $0 \le \widehat{\imath}_{k-1} < n_{k-1}$, we get

$$\prod_{j=0}^{k-1} U_j^{a_j} = \theta_{k-1} U_k^{\widehat{\imath}_{k-1}} \prod_{j=0}^{k-2} U_j^{a'_j}$$

for some $a'_j \in \mathbf{N}$. We finally get by induction that

$$\prod_{j=0}^{k} U_j^{i_j} = \theta_I T^{r_I} \prod_{0}^{k} U_j^{\widehat{\imath}_j}$$

in $\mathrm{gr}_\nu R_\nu$, with $\theta_I \in \mathbf{C}^*$, $r_I (= r_k) \ge 0$, $0 \le \widehat{\imath}_j < n_j$ for $1 \le j \le k$ and $\widehat{\imath}_0 \ge 0$.

Now $\sum_0^k i_j\tilde\beta_j = \nu(\phi)$ and $\nu(T) = 0$, hence $\sum_0^k \widehat{\imath}_j\tilde\beta_j = \nu(\phi)$. Suppose $\sum_0^k \widehat{\imath}_j\tilde\beta_j = \sum_0^k \tilde\imath_j\tilde\beta_j$ with $0 \le \tilde\imath_j < n_j$ for $1 \le j \le k$. Since $|\widehat{\imath}_k - \tilde\imath_k| < n_k$, the definition of n_k gives $\widehat{\imath}_k = \tilde\imath_k$. By induction we get $\widehat{\imath}_j = \tilde\imath_j$ for all $j \ge 0$. We have proved that ϕ can be written in the form (3.2).

Uniqueness of both decompositions (3.1), (3.2) comes from unique factorization in $\mathrm{gr}_\nu\mathbf{C}(x)[y]$. Indeed, when $n_k = \infty$, $i_k = \delta_k(\phi)$. From $\sum i_j\tilde\beta_j = \nu(\phi)$ and $i_j < n_j$ for $1 \le j < k$ we deduce uniqueness of the decomposition (3.1). When $n_k < \infty$, $\nu(\phi)$ determines i_j for all $j \ge 0$, and the polynomial $p(T)$ is determined as ϕ admits a unique decomposition into prime factors (see Corollary 3.2). This concludes the proof of the theorem. □

3.2. **Homogeneous decomposition II.** We now turn to infinite STKP's.

Theorem 3.3. *Let* $\nu := \mathrm{val}\,[(U_j)_0^\infty; (\tilde\beta_j)_0^\infty]$. *Pick* $0 \ne \phi \in R$.

(i) *If* $n_j \ge 2$ *for infinitely many* j, *then there exists* $k = k(\phi)$ *such that*

$$\phi = \alpha \prod_{j=0}^{k} U_j^{i_j} \text{ in } \mathrm{gr}_\nu R \tag{3.4}$$

with $\alpha \in \mathbf{C}^*$, $0 \le i_j < n_j$ *for* $1 \le j \le k$, *and* $i_0 \ge 0$.

(ii) *If* $n_j = 1$ *for* $j \gg 1$, *then there exists* $k = k(\phi)$ *such that*

$$\phi = \alpha \cdot U_\infty^n \cdot \prod_{j=0}^{k} U_j^{i_j} \text{ in } \mathrm{gr}_\nu R \tag{3.5}$$

with $\alpha \in \mathbf{C}$, $0 \le i_j < n_j$ *for* $1 \le j \le k$, $i_0 \ge 0$ *and* $n \ge 0$.

Both decompositions (3.4), (3.5) *are unique.*

The analogue of Corollary 3.2 is

Corollary 3.4. *Assume* ν *is associated to an infinite STKP.*

(i) *When* $n_j \ge 2$ *for infinitely many* j, *the ring* $\mathrm{gr}_\nu \mathbf{C}(x)[y]$ *is a field.*
(ii) *When* $n_j = 1$ *for* $j \gg 1$, *the only irreducible element of* $\mathrm{gr}_\nu \mathbf{C}(x)[y]$ *is* U_∞.

Proof of Theorem 3.3. Fix $\phi \in R$, and suppose it is in Weierstrass form, polynomial in y. First assume that $n_j \ge 2$ for infinitely many j. For $j \gg 1$ $\deg_y \phi < \deg_y U_j$, hence $\phi = \alpha \prod_{l=0}^{j-1} U_l^{i_l}$ in $\mathrm{gr}_{\nu_j} R$ for some $\alpha \in \mathbf{C}^*$, and $i_l < n_l$ for $l \ge 1$ (see the beginning of the proof of Theorem 3.1). As $\nu \ge \nu_j$, we get $\phi = \alpha \prod_{l=0}^{j-1} U_l^{i_l}$ also in $\mathrm{gr}_\nu R$.

When $n_j = 1$ for $j \gg 1$, write $\phi = U_\infty^n \phi'$ with ϕ', U_∞ prime. For $j \gg 1$ $\nu_{j+1}(\phi') = \nu_j(\phi') = \nu(\phi')$. Hence $\phi' = \alpha \prod_{l=0}^{j-1} U_l^{i_l}$ in $\mathrm{gr}_{\nu_j} R$ and $\mathrm{gr}_\nu R$ as above. ☐

Proof of Corollary 3.4. The first assertion is immediate by Lemma 2.15. Assume $n_j = 1$ for $j \gg 1$ and pick $\phi \in R$. If U_∞ does not divide ϕ, then (3.4) holds. For $k \gg 1$ we get $\delta_k(\phi) = 0$. Thus ϕ is invertible in $\mathrm{gr}_{\nu_k} \mathbf{C}(x)[y]$, hence in $\mathrm{gr}_\nu \mathbf{C}(x)[y]$.

On the other hand $U_\infty = \phi\psi$ in $\mathrm{gr}_\nu \mathbf{C}(x)[y]$ implies either $\nu(\phi) = \infty$ or $\nu(\psi) = \infty$. Hence U_∞ divides either ϕ or ψ as formal power series hence in $\mathrm{gr}_\nu \mathbf{C}(x)[y]$. We conclude noting that U_∞ cannot be a unit in $\mathrm{gr}_\nu \mathbf{C}(x)[y]$. Indeed $\nu(U_\infty \phi) = \nu(1)$ implies that $U_\infty \phi$ is prime with U_∞. ☐

3.3. **Numerical invariants.** Knowing the structure of the graded rings allows us to compute the numerical invariants introduced in Section 1.2.

Theorem 3.5. *Let* $\nu := \mathrm{val}\,[(U_j)_0^k; (\tilde\beta_j)_0^k]$, $1 \le k < \infty$.

(i) *If* $\tilde\beta_0 = \infty$ *or* $\tilde\beta_k = \infty$, *then* $\mathrm{rk}(\nu) = \mathrm{rat.rk}(\nu) = 2$ *and* $\mathrm{tr.deg}(\nu) = 0$.
(ii) *If* $\tilde\beta_0 < \infty$ *and* $\tilde\beta_k \in \mathbf{Q}^*$, *then* $\mathrm{rk}(\nu) = 1$, $\mathrm{rat.rk}(\nu) = 1$ *and* $\mathrm{tr.deg}(\nu) = 1$.
(iii) *If* $\tilde\beta_0 < \infty$ *and* $\tilde\beta_k \notin \mathbf{Q}^*$, *then* $\mathrm{rk}(\nu) = 1$, $\mathrm{rat.rk}(\nu) = 2$ *and* $\mathrm{tr.deg}(\nu) = 0$.

Theorem 3.6. *Let* $\nu := \mathrm{val}\,[(U_j); (\tilde\beta_j)]$ *be defined by an infinite STKP.*

(i) *If* $n_j \ge 2$ *for infinitely many* j, *then* $\mathrm{rk}(\nu) = \mathrm{rat.rk}(\nu) = 1$, $\mathrm{tr.deg}(\nu) = 0$.
(ii) *If* $n_j = 1$ *for* $j \gg 1$, *then* $\mathrm{rk}(\nu) = \mathrm{rat.rk}(\nu) = 2$, $\mathrm{tr.deg}(\nu) = 0$.

Before proving these theorems we introduce some terminology.

Definition 3.7. Consider a centered valuation ν on R. We say that ν is

(i) *toroidal* if $\mathrm{rk}(\nu) = 1$, $\mathrm{rat.rk}(\nu) + \mathrm{tr.deg}(\nu) = 2$;
(ii) *divisorial* if $\mathrm{rk}(\nu) = \mathrm{rat.rk}(\nu) = \mathrm{tr.deg}(\nu) = 1$;
(iii) *irrational* if $\mathrm{rk}(\nu) = 1$, $\mathrm{rat.rk}(\nu) = 2$, $\mathrm{tr.deg}(\nu) = 0$;
(iv) *infinitely singular* if $\mathrm{rk}(\nu) = \mathrm{rat.rk}(\nu) = 1$, $\mathrm{tr.deg}(\nu) = 0$;
(v) a *curve valuation* if $\mathrm{rk}(\nu) = \mathrm{rat.rk}(\nu) = 2$, $\mathrm{tr.deg}(\nu) = 0$;

Remark 3.8. Theorems 3.5 3.6 and 4.1 below show that any centered valuation is of exactly one of types (ii)-(v). Definition (ii) of divisorial valuations is consistent with (E2) in Section 1.3 (see [ZS]). The same is true for curve valuations, as follows from the analysis in Section 1.4, and for toroidal valuations. Infinitely singular valuations are characterized by the property that any (weakly) approximating sequence of curve valuations have multiplicity tending to infinity.

Proof of Theorem 3.5. The two invariants $\mathrm{rk}(\nu)$ and $\mathrm{rat.rk}(\nu)$ can be read directly from the value group $\nu(R)$ so their computation is straightforward. For the computation of $\mathrm{tr.deg}(\nu)$ we rely on Theorem 3.1 and proceed as follows.

When $\tilde\beta_0 = \infty$, $\tilde\beta_k = \infty$ or $\tilde\beta_k \notin \mathbf{Q}^*$ (so $\tilde\beta_k \notin \sum_0^{k-1} \mathbf{Z}\tilde\beta_j$), then (3.1) applies. Pick $\phi, \psi \in R$ with $\nu(\phi) = \nu(\psi)$. Then $\phi = \alpha \prod_{l=0}^k U_l^{i_l}$, $\psi = \gamma \prod_{l=0}^k U_l^{j_l}$ modulo ν, with $\sum_0^k i_l\tilde\beta_l = \sum_0^k j_l\tilde\beta_l$. Hence $i_k = j_k$ since $n_k = \infty$. Now $\sum_0^{k-1} i_l\tilde\beta_l = \sum_0^{k-1} j_l\tilde\beta_l$ and $0 \le i_l, j_l < n_l$ for all $1 \le j < k$ so $i_l = j_l$ for all l by (2.2). Hence $\phi/\psi = \alpha/\gamma \in \mathbf{C}^*$ in $\mathrm{gr}_\nu R_\nu$. We have shown that $k_\nu := R_\nu/\mathfrak{m}_\nu \cong \mathbf{C}$ so $\mathrm{tr.deg}(\nu) = 0$.

If $\tilde\beta_0 < \infty$ and $\tilde\beta_k \in \mathbf{Q}^*$, then $n_k < \infty$. For $\phi, \psi \in R$ with $\nu(\phi) = \nu(\psi)$ write

$$\phi = p(T) \prod_{l=0}^k U_l^{i_l} \quad \text{and} \quad \psi = q(T) \prod_{l=0}^k U_l^{j_l} \quad \text{in } \mathrm{gr}_\nu R_\nu$$

as in (3.2), with $0 \le i_l, j_l < n_l$ for $1 \le l \le k$, $i_0, j_0 \ge 0$ and $p, q \in \mathbf{C}[T]$. As $\sum_0^k i_l\tilde\beta_l = \sum_0^k j_l\tilde\beta_l$, one has $i_l = j_l$ for all l. Hence $\phi/\psi = p(T)/q(T)$ in $\mathrm{gr}_\nu R_\nu$. This shows $k_\nu \cong \mathbf{C}(T)$ so $\mathrm{tr.deg}(\nu) = 1$. □

Proof of Theorem 3.6. First assume $n_j \ge 2$ for infinitely many j. Set $\nu_k := \mathrm{val}\,[(U_j)_0^k; (\tilde\beta_j)_0^k]$. Then the sequence $\nu_k(\phi)$ is eventually stationary for any $\phi \in R$. One easily checks that $\mathrm{rk}(\nu) = \mathrm{rat.rk}(\nu) = 1$. Let us compute $\mathrm{tr.deg}(\nu)$. Fix $\phi, \psi \in R$ with $\nu(\phi) = \nu(\psi)$. For $k \gg 1$ we have $\nu_k(\phi) = \nu(\phi) = \nu(\psi) = \nu_k(\psi)$. Write $\phi = \alpha \prod_{l=0}^k U_l^{i_l}$ and $\psi = \gamma \prod_{l=0}^k U_l^{j_l}$ where $\alpha, \gamma \in \mathbf{C}^*$, $0 \le i_l, j_l < n_l$ for $l \ge 1$ and apply the preceding arguments. We get $\phi/\psi = \alpha/\gamma$ in $\mathrm{gr}_{\nu_k} R_{\nu_k}$ hence in $\mathrm{gr}_\nu R_\nu$. This shows $k_\nu \cong \mathbf{C}$.

We leave to the reader the last case when $n_j = 1$ for $j \gg 1$. □

4. From valuations to STKP's

We now show that every centered valuation on R is represented by an STKP.

Theorem 4.1. *For any centered valuation ν on R, there exists a unique STKP $[(U_j)_0^n; (\tilde\beta_j)_0^n]$, $1 \le n \le \infty$, such that $\nu = \mathrm{val}\,[(U_j); (\tilde\beta_j)]$. We have $\nu(U_j) = \tilde\beta_j$ for all j. Further, if $k < n$ and $\nu_k := \mathrm{val}\,[(U_j)_0^k; (\tilde\beta_j)_0^k]$, then $\nu(\phi) \ge \nu_k(\phi)$ for all $\phi \in R$ and $\nu(\phi) > \nu_k(\phi)$ if and only if U_{k+1} divides ϕ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.*

Remark 4.2. If ν is a Krull valuation, the same result holds for a Γ-STKP.

Remark 4.3. In Spivakovsky's terminology [Sp], the subset of (U_j) for which $n_j > 1$ forms a minimal generating sequence for the valuation ν.

Proof of Theorem 4.1. We construct by induction on k a valuation ν_k so that $\nu_k(U_j) = \tilde{\beta}_j$ for $j \leq k$. Let $U_0 = x$, $U_1 = y$ and $\tilde{\beta}_0 = \nu(x)$, $\tilde{\beta}_1 = \nu(y)$. Assume $\nu_k := \mathrm{val}\,[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ has been defined. By Theorem 2.7, $\nu(\phi) \geq \nu_k(\phi)$ for $\phi \in R$. As $\nu(x) = \nu_k(x)$ this also holds for $\phi \in \mathbf{C}(x)[y]$. If $\nu = \nu_k$, then we are done. If not, set $\mathcal{D}_k := \{\phi \in \mathbf{C}(x)[y] \mid \nu(\phi) > \nu_k(\phi)\}$.

Lemma 4.4. *$\mathcal{D}_k$ defines a prime ideal in* $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$.

Proof. Let us check that $\mathcal{D}_k$ is well defined in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$. Pick $\phi, \phi' \in \mathbf{C}(x)[y]$ with $\phi = \phi'$ modulo ν_k. If $\phi \in \mathcal{D}_k$ but $\phi' \notin \mathcal{D}_k$, then $\nu(\phi') = \nu_k(\phi') = \nu_k(\phi) < \nu(\phi)$. So $\nu(\phi - \phi') = \nu(\phi') = \nu_k(\phi') < \nu_k(\phi - \phi')$, a contradiction. To show that $\mathcal{D}_k$ is a prime ideal is easy and left to the reader. □

We continue the proof of the theorem. Recall that $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ is a Euclidean domain. Hence Corollary 3.2 and Lemma 2.6 show that $\mathcal{D}_k$ is generated by a unique irreducible element $U_{k+1} = U_k^{n_k} - \theta_k \prod_{j=0}^{k-1} U_j^{m_{k,j}}$ with $\theta_k \in \mathbf{C}^*$, $0 \leq m_{k,j} < n_j$ for $j \geq 1$ and $m_{k,0} \geq 0$. Define $\tilde{\beta}_{k+1} := \nu(U_{k+1})$. It is easy to check that $[(U_j)_0^{k+1}; (\tilde{\beta}_j)_0^{k+1}]$ is an STKP. This completes the induction step.

Either the induction terminates at some finite k, or we get an infinite STKP $[(U_j); (\beta_j)]$. In the latter case we claim that $\nu = \mathrm{val}\,[(U_j); (\beta_j)]$. This amounts to showing that if $\phi \in R$, then the increasing sequence $\nu_k(\phi)$ converges to $\nu(\phi)$.

If $\nu_k(\phi) = \nu(\phi)$ for $k \gg 1$ then we are done, so assume $\nu_k(\phi) < \nu(\phi)$ for all k. Then $\phi \in \mathcal{D}_k$ so U_{k+1} divides ϕ in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$. In particular, $\nu_{k+1}(\phi) > \nu_k(\phi)$, and $\deg_y(\phi) \geq \deg_y(U_{k+1}) = d_{k+1}$. Hence d_k is bounded, and $n_k = 1$ for $k \gg 1$. By Theorem 2.21, $U_k \to U_\infty$ in R. Moreover U_∞ divides ϕ in R or else $\nu_k(\phi)$ would be stationary for k large. But then $\nu(\phi) \geq \lim \nu_k(\phi) = \infty$ so $\nu_k(\phi) \to \nu(\phi)$. This completes the proof of the theorem. □

5. A computation

We now compute $\nu(\phi)$ for a normalized valuation ν and $\phi \in \mathfrak{m}$ irreducible in terms of STKP's. This computation will be crucial to describe the metric tree structure of valuation space; see Proposition 8.3.

Let ν_ϕ be the curve valuation associated to ϕ. Write $\nu = \mathrm{val}\,[(U_j); (\tilde{\beta}_j)]$ and $\nu_\phi = \mathrm{val}\,[(U_j^\phi); (\tilde{\beta}_j^\phi)]$. Assume $\nu \neq \nu_\phi$ and define the contact order of ν and ν_ϕ by

$$\mathrm{con}\,(\nu, \nu_\phi) = \max\{j \mid U_j = U_j^\phi\}. \tag{5.1}$$

Let n_j^ϕ be the integers defined by (2.2) for ν_ϕ and set $\gamma_k^\phi = \prod_{j \geq k} n_j^\phi$ for $k \geq 1$. These products are in fact finite as $n_j^\phi = 1$ for $j \gg 1$.

Proposition 5.1. *If $\phi = x$ (up to a unit), then $\nu(\phi) = \tilde{\beta}_0$. Otherwise*

$$\nu(\phi) = \gamma_k^\phi \min\{\tilde{\beta}_k, \tilde{\beta}_k^\phi\} \min\{1, \tilde{\beta}_0 / \tilde{\beta}_0^\phi\}, \tag{5.2}$$

where $k = \mathrm{con}\,(\nu, \nu_\phi)$.

Later on we will be interested in the quotient $\nu(\phi)/m(\phi)$. Applying Proposition 5.1 to $\nu = \nu_\mathfrak{m}$ we obtain $m(\phi) = \gamma_1^\phi / \tilde{\beta}_0^\phi$ if $\phi \neq x$; this leads to

Proposition 5.2. *If $\nu \in \mathcal{V}$ and $\phi \in \mathfrak{m}$ is irreducible, then*

$$\frac{\nu(\phi)}{m(\phi)} = d_k^{-1} \min\{\tilde\beta_k, \tilde\beta_k^\phi\} \min\{\tilde\beta_0, \tilde\beta_0^\phi\},$$

where $k = \mathrm{con}\,(\nu, \nu_\phi)$ and $d_k = \deg_y U_k = \prod_1^{k-1} n_j$.

Proof of Proposition 5.1. The case $\phi = x$ is trivial, so assume $\phi \neq x$, i.e. $\tilde\beta_0^\phi < \infty$. We may then assume $\phi = U_l^\phi$, where $l := \mathrm{length}\,\nu_\phi \in [k, \infty]$. Note that $\min\{\tilde\beta_0, \tilde\beta_1\} = \min\{\tilde\beta_0^\phi, \tilde\beta_1^\phi\} = 1$ since ν and ν_ϕ are normalized.

Let us first consider the case when $\tilde\beta_0 = \tilde\beta_0^\phi$ and $\tilde\beta_1 = \tilde\beta_1^\phi$; this holds e.g. if $k \geq 2$. Then $\tilde\beta_j = \tilde\beta_j^\phi$ for $0 \leq j < k$. If $l = k$, then $\phi = U_k$ so $\nu(\phi) = \tilde\beta_k$ and $\tilde\beta_k^\phi = \infty$, which implies (5.2). Hence assume $l > k$. Define $\xi_j = \tilde\beta_{k+j}^\phi$ and $\eta_j = \min\{\tilde\beta_k, \tilde\beta_k^\phi\}\gamma_k^\phi/\gamma_{k+j}^\phi$ for $0 \leq j \leq l-k$. Then $\nu_\phi(U_{k+j}^\phi) = \xi_j$ for $j \geq 0$ and $\nu(U_j^\phi) = \tilde\beta_j$ for $0 \leq j \leq k$. We will prove inductively that $\nu(U_{k+j}^\phi) = \eta_j$ for $j \geq 1$; when $j = l-k$ this gives (5.2). The induction is based on (2.3), which reads

$$U_{k+j+1}^\phi = (U_{k+j}^\phi)^{n_{k+j}^\phi} - \theta_{k+j}^\phi \prod_{i=0}^{k+j-1} (U_i^\phi)^{m_{k+j,i}^\phi} =: A_j - B_j. \tag{5.3}$$

Let us first show that $\nu(U_{k+1}^\phi) = \eta_1$, using (5.3) for $j = 0$. There are three cases, depending on $\tilde\beta_k$ and $\tilde\beta_k^\phi$. The first case is when $\tilde\beta_k > \tilde\beta_k^\phi$. Then $n_k^\phi\tilde\beta_k > n_k^\phi\tilde\beta_k^\phi = \sum_0^{k-1} m_{k,i}^\phi\tilde\beta_i$, so using (5.3) we obtain $\nu(U_{k+1}^\phi) = \tilde\beta_k^\phi n_k^\phi = \eta_1$. Similarly, in the second case, $\tilde\beta_k < \tilde\beta_k^\phi$, then $\sum_0^{k-1} m_{k,i}^\phi\tilde\beta_i = n_k^\phi\tilde\beta_k^\phi > n_k^\phi\tilde\beta_k$, so that $\nu(U_{k+1}^\phi) = n_k^\phi\tilde\beta_k = \eta_1$. The third case is when $\tilde\beta_k = \tilde\beta_k^\phi$. Set $\nu_k := \mathrm{val}\,[(U_j)_0^k; (\tilde\beta_j)_0^k]$. Then U_{k+1}^ϕ is irreducible in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$ and $\nu_k(U_{k+1}^\phi) = \tilde\beta_k n_k^\phi = \eta_1$. If $\mathrm{length}\,\nu = k$ then $\nu = \nu_k$ and $\nu(U_{k+1}^\phi) = \eta_1$ so assume $\mathrm{length}\,\nu > k$. By assumption $U_{k+1}^\phi \neq U_{k+1}$, hence U_{k+1}^ϕ does not belong to the ideal generated by U_{k+1} in $\mathrm{gr}_{\nu_k}\mathbf{C}(x)[y]$. But this ideal coincides with $\{\nu > \nu_k\}$ by Theorem 4.1, so $\nu(U_{k+1}^\phi) = \nu_k(U_{k+1}^\phi) = \eta_1$.

Now fix $1 \leq j < l-k$ and assume that $\nu(U_{k+i}^\phi) = \eta_i$ for $1 \leq i \leq j$. We will prove that $\nu(U_{k+j+1}^\phi) = \eta_{j+1}$ using (5.3). Write $a_i = m_{k+j,k+i}^\phi$ for $0 \leq i < j$ and $c = n_{k+j}^\phi$. Then $\nu(A_j) = c\eta_j$ and

$$\begin{aligned}\nu(B_j) = \sum_{i=1}^{k} m_{k+j,i}^\phi\tilde\beta_i + \sum_{i=1}^{j-1} a_i\eta_i &= \sum_{i=1}^{k+j-1} m_{k+j,i}^\phi\tilde\beta_i^\phi + a_0(\tilde\beta_k - \xi_0) + \sum_{i=1}^{j-1} a_i(\eta_i - \xi_i)\\ &\geq \sum_{i=1}^{k+j-1} m_{k+j,i}^\phi\tilde\beta_i^\phi + \sum_{i=0}^{j-1} a_i(\eta_i - \xi_i) = c\xi_j + \sum_{i=0}^{j-1} a_i(\eta_i - \xi_i).\end{aligned}$$

To show that $\nu(U_{k+j+1}^\phi) = \eta_{j+1}$ we only need to show $\nu(B_j) > c\eta_j$ or simply $\sum_0^{j-1} a_i(\xi_i - \eta_i) > c(\xi_j - \eta_j)$. Note that $\sum_0^{j-1} a_i\xi_i \leq c\xi_j$ and that the sequence

$(\eta_i/\xi_i)_0^j$ is strictly decreasing. Set $p_i = a_i\xi_i/\sum_0^{j-1} a_i\xi_i$. Then $\sum_0^{j-1} p_i = 1$ so

$$\sum_{i=0}^{j-1} a_i(\xi_i - \eta_i) = \left(\sum_{i=0}^{j-1} a_i\xi_i\right)\sum_{i=0}^{j-1} p_i\left(1 - \frac{\eta_i}{\xi_i}\right) < c\xi_j\left(1 - \frac{\eta_j}{\xi_j}\right) = c(\xi_j - \eta_j).$$

This completes the proof when $\tilde\beta_0 = \tilde\beta_0^\phi$ and $\tilde\beta_1 = \tilde\beta_1^\phi$. The remaining cases all have $k = 1$ and are as follows: $\tilde\beta_0 \geq \tilde\beta_1 = 1$ and $1 = \tilde\beta_0^\phi \leq \tilde\beta_1^\phi$; $\tilde\beta_0 \geq \tilde\beta_1 = 1$ and $\tilde\beta_0^\phi \geq \tilde\beta_1^\phi = 1$; $1 = \tilde\beta_0 \leq \tilde\beta_1$ and $1 = \tilde\beta_0^\phi < \tilde\beta_1^\phi$; $1 = \tilde\beta_0 \leq \tilde\beta_1$ and $\tilde\beta_0^\phi \geq \tilde\beta_1^\phi = 1$. These are handled in the same way as above. The details are left to the reader. □

Part 3. Tree structures

In this third part of the paper we study tree structures on valuation space $\mathcal{V}$. We will use the term "tree" instead of "**R**-tree" even though all trees will be modeled on the real line. We first describe a notion of tree in terms of a partially ordered set. This notion seems to be new, although related to the approach of Berkovich [B]. A natural weak tree topology is defined from the ordering. We then recall the definition of a metric tree (typically called **R**-tree in the literature [MO]). In both cases, a tree is a one-dimensional object where any two points are connected by a unique arc or segment.

In Section 7, we show that the natural order on $\mathcal{V}$ induces a (non-metric) tree structure, which we describe in some detail. We then define in Section 8 a numerical invariant of a valuation, its skewness. This leads to a natural parameterization of $\mathcal{V}$, and to a natural metric turning $\mathcal{V}$ into a metric tree.

6. Trees

6.1. **Trees.** A partially ordered set $(\mathcal{T}, \leq)$ is a *tree* if:

(T1) $\mathcal{T}$ has a unique minimal element σ_0, called the *root* of $\mathcal{T}$;

(T2) every full, totally ordered subset of T is isomorphic to a real interval.

Here a totally ordered subset $\mathcal{S} \subset \mathcal{T}$ is *full* if $\sigma, \sigma' \in \mathcal{S}$, $\sigma'' \in \mathcal{T}$ and $\sigma \leq \sigma'' \leq \sigma'$ implies $\sigma'' \in \mathcal{S}$. Statement (T2) asserts that there exists an order preserving bijection of any such $\mathcal{S}$ onto a real interval.

Remark 6.1. More generally, if Λ is a totally ordered set, then a Λ-tree is a partially ordered set such that (T1) and (T2) hold, with the interval in (T2) being an interval in Λ. Interesting examples include $\Lambda = \mathbf{Z}$ and $\Lambda = \mathbf{N} \cup \{\infty\}$.

It follows from the completeness of $\mathbf{R}$ that every subset $S \subset \mathcal{T}$ admits an infimum, denoted by $\wedge_{\tau \in S}\tau$. Indeed, the set $\{\sigma \in \mathcal{T} \mid \sigma \leq \tau \ \forall \tau \in S\}$ is isomorphic to the intersection of closed real intervals with common left endpoint.

Any point in a tree can serve as a root. Indeed, if $(\mathcal{T}, \leq)$ is a tree rooted at σ_0 and σ_0' is any point in $\mathcal{T}$, then we may define a new partial ordering $\leq'$ on $\mathcal{T}$ by declaring $\sigma_1 \leq' \sigma_2$ iff

$$\sigma_0' \leq \sigma_1 \leq \sigma_2; \quad \text{or} \quad \sigma_0' \geq \sigma_1 \geq \sigma_2 \quad \text{or} \quad \sigma_0' \geq \sigma_1, \ \sigma_1 \not\leq \sigma_2, \ \sigma_1 \not\geq \sigma_2.$$

Then $(\mathcal{T}, \leq')$ is a tree rooted at σ_0'. As is easily verified, none of the following constructions is dependent on the choice of root.

If σ_1, σ_2 are two points in $\mathcal{T}$, then we set

$$[\sigma_1, \sigma_2] := \{\sigma \in \mathcal{T} \mid \sigma_1 \wedge \sigma_2 \leq \sigma \leq \sigma_1 \quad \text{or} \quad \sigma_1 \wedge \sigma_2 \leq \sigma \leq \sigma_2\}.$$

We call $[\sigma_1, \sigma_2]$ a *segment*, and define $[\sigma_1, \sigma_2[:= [\sigma_1, \sigma_2] \setminus \{\sigma_2\}$ and, similarly, $]\sigma_1, \sigma_2]$, $]\sigma_1, \sigma_2[$.

Two trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are *isomorphic* if there exists an order-preserving bijection, or *tree isomorphism* $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$.

Given a point $\sigma \in \mathcal{T}$ we define an equivalence relation on the set of nonempty segments $]\sigma, \sigma']$ in $\mathcal{T}$ by declaring two segments to be equivalent if they intersect. An equivalence class is called a *tangent vector* at σ and the set of tangent vectors is called the *tangent space* at σ, denoted $T_\sigma\mathcal{T}$. (The tangent spaces should be thought of as projectivized.) We say that the segment $]\sigma, \sigma']$ (or simply the point σ') *represents* the tangent vector.

A point in $\mathcal{T}$ is an *end* if its tangent space has only one element. It is a *regular point* if the tangent space has two elements, and a *branch point* otherwise.

A subset $\mathcal{S} \subset \mathcal{T}$ is a *subtree* if the segment $[\sigma_1, \sigma_2]$ is in $\mathcal{S}$ for every $\sigma_1, \sigma_2 \in \mathcal{S}$. Clearly $\mathcal{S}$ is a tree in the induced partial order (rooted at any point in $\mathcal{S}$).

A tree $\mathcal{T}$ is *complete* if every increasing sequence $(\sigma_i)_{i\geq 1}$ in $\mathcal{T}$ has a majorant, i.e. an element $\sigma_\infty \in \mathcal{T}$ with $\sigma_i \leq \sigma_\infty$ for every i. Any tree $\mathcal{T}$ has a *completion* $\bar{\mathcal{T}}$ obtained by adding points corresponding to unbounded increasing sequences σ_i in $\mathcal{T}$. All points in $\bar{\mathcal{T}} \setminus \mathcal{T}$ are ends. We write $\mathcal{T}^e$ for the set of ends in $\bar{\mathcal{T}}$.

A *parameterization* of a tree $\mathcal{T}$ is an increasing mapping $\alpha : \mathcal{T} \to [0, \infty)$ whose restriction to any full, totally ordered subset of $\mathcal{T}$ gives a bijection onto a real interval. The tree is then *parameterizable*. It is convenient—and equivalent—to allow parameterizations with values either in $[0, \infty]$ or in $[0, 1]$. Thus a tree $\mathcal{T}$ is parameterizable iff its completion $\bar{\mathcal{T}}$ is. We do not know if there exists a non parameterizable tree. All trees we consider in this paper will be parameterizable.

6.2. **The weak topology.** A tree carries a natural *weak topology* defined as follows. If $\vec{v} \in T_\sigma\mathcal{T}$ is a tangent vector at a point $\sigma \in \mathcal{T}$, then set

$$V_{\vec{v}} = \{\tau \in \mathcal{T} \mid \text{the segment }]\sigma, \tau] \text{ represents } \vec{v}\}.$$

Then the weak topology is generated by the sets $V_{\vec{v}}$ (i.e. the open sets are unions of finite intersections of such sets). This topology is Hausdorff. Any complete subtree $\mathcal{S}$ of $\mathcal{T}$ is weakly closed in $\mathcal{T}$ and the injection $\mathcal{S} \to \mathcal{T}$ is an embedding. In particular, any segment $\gamma = [\sigma, \sigma']$ in $\mathcal{T}$ is closed, and the induced topology on γ coincides with the standard topology on $[0, 1]$ under the identification of γ with the latter set. However, the branching of a tree does play an important role for the weak topology:

Proposition 6.2. *If $\mathcal{T}$ is a tree and $\sigma_n, \sigma \in \mathcal{T}$, then σ_n converges weakly to σ iff for every tangent vector $\vec{v} \in T_\sigma\mathcal{T}$ such that the set $I_{\vec{v}} := \{n \mid \sigma_n \text{ represents } \vec{v}\}$ is infinite, the segments $]\sigma, \sigma_n]$, $n \in I_{\vec{v}}$ have empty intersection. In particular, if $I_{\vec{v}}$ is finite for every $\vec{v}$, then $\sigma_n \to \sigma$.*

We leave the proof to the reader as we will not use this result. A fundamental property of the weak topology is given by

Proposition 6.3. *Any parameterizable complete tree is weakly compact.*

Sketch of proof. Consider a parameterization $\alpha : \mathcal{T} \to [0, 1]$ of a complete tree $(\mathcal{T}, \leq)$. Let $E = [0, 1]^{\mathcal{T}}$, which is compact in the product topology. Define a mapping $j : \mathcal{T} \to E$ by $j(\sigma)(\tau) = \alpha(\sigma \wedge \tau)$. One shows (with some work) that j is an embedding of $\mathcal{T}$ into E and that $j(\mathcal{T})$ is closed. Thus $\mathcal{T}$ is homeomorphic to a closed subset of a compact space, hence is itself compact. □

Note that a tree $\mathcal{T}$ which is not complete cannot be weakly compact: a sequence in $\mathcal{T}$ that increases to an element in $\bar{\mathcal{T}} \setminus \mathcal{T}$ cannot have a convergent subsequence.

6.3. **Metric trees.** A *metric tree* is a metric space $\mathcal{T}$ in which every two points is joined by a unique arc, or segment, and this segment is isometric to a real interval. It is known [MO] that a metric space is a metric tree (i.e. admits a compatible metric under which it becomes a metric tree) iff it is uniquely pathwise connected and locally pathwise connected.

Any metric tree gives rise to a tree in the previous sense. Indeed, fix $\sigma_0 \in \mathcal{T}$ (the root) and define a partial ordering on $\mathcal{T}$: $\sigma \leq \sigma'$ iff σ belongs to the segment between σ_0 and σ'. Clearly (T1) and (T2) hold. We say that the metric is *compatible* with the nonmetric tree structure. Notice that $\mathcal{T}$ is naturally parameterized by setting $\alpha(\sigma) = d(\sigma_0, \sigma)$.

Conversely, consider a nonmetric tree $\mathcal{T}$ with a parameterization $\alpha : \mathcal{T} \to [0, 1]$. Define $d : \mathcal{T} \times \mathcal{T} \to [0, 2]$ by $d(\sigma, \tau) = (\alpha(\sigma) - \alpha(\sigma \wedge \tau)) + (\alpha(\tau) - \alpha(\sigma \wedge \tau))$.

Lemma 6.4. *$(\mathcal{T}, d)$ is a metric tree.*

Proof. It is straightforward to verify that $(\mathcal{T}, d)$ is a metric space in which every two points is joined by an arc isometric to a real interval. What remains to be seen is that there is a *unique* arc between any two points in $\mathcal{T}$. For this, consider a continuous injection $\imath : I \to (\mathcal{T}, d)$ of a real interval $I = [t_1, t_2]$ into $\mathcal{T}$, and let $\sigma_i = \imath(t_i)$. By declaring σ_1 to be the root of $\mathcal{T}$, we may suppose $\sigma_1 \leq \sigma_2$. Define $\pi(t) = \imath(t) \wedge \sigma_2$, and suppose $\pi^{-1}\{t\}$ has an interior point for some $t \in I$. If $[a, b]$, $a < b$ denotes a non trivial connected component of $\pi^{-1}\{t\}$, the tree structure of $\mathcal{T}$ implies $\imath(a) = \imath(b)$. This contradicts the injectivity of $\imath$. From the fact that the preimage by π of any point in $[\sigma_1, \sigma_2]$ has empty interior, we infer that $\imath$ maps I into the segment $[\sigma_1, \sigma_2]$. The injectivity of $\imath$ then gives that $\imath$ is an increasing homeomorphism of I onto $[\sigma_1, \sigma_2]$. This completes the proof. □

As the following example shows, two metric trees may be isomorphic as trees without being homeomorphic as topological spaces.

Example 6.5. Fix a set X and set $\mathcal{T} = X \times \mathbf{R}/\sim$, where $(x, 0) \sim (y, 0)$ for any $x, y \in X$. Then $\mathcal{T}$ is a tree under the partial ordering $\leq$ defined by $(x, s) \leq (y, t)$ iff either $s = t = 0$, or $x = y$ and $s \leq t$.

Given $g : X \to (0, \infty)$ define a metric d_g on $\mathcal{T}$ by $d_g((x, s), (x, t)) = g(x)|s - t|$ and $d_g((x, s), (y, t)) = g(x)s + g(y)t$ if $x \neq y$. Then $(\mathcal{T}, d_g)$ is homeomorphic to $(\mathcal{T}, d_h)$ iff there exists $C > 1$ such that $g(x)/h(x) \in [1/C, C]$ for all x.

If a metric tree $\mathcal{T}$ is complete as a tree, then $(\mathcal{T}, d)$ is a complete metric space for every tree metric d on $\mathcal{T}$. As a partial converse, if a nonmetric tree $\mathcal{T}$ admits a compatible, complete tree metric d of finite diameter, then $\mathcal{T}$ is complete as a tree. Notice, however, that $\mathbf{R}$ with its standard metric is complete as a metric space but not as a tree.

If a metric tree $\mathcal{T}$ has finite diameter, then the completion of $\mathcal{T}$ as a metric space agrees with the completion of $\mathcal{T}$ as a tree. Moreover, it is always possible to find an equivalent metric in which $\mathcal{T}$ has finite diameter.

A metric tree admits two natural topologies, the weak topology as a (non-metric) tree and the topology as a metric space, which we will also refer to as the *strong topology*. As we noted above, the latter depends on the choice of metric.

Proposition 6.6. *The strong topology on a metric tree $\mathcal{T}$ is at least as strong as the weak topology.*

Proof. Let σ_n be a sequence of points in the metric tree $(\mathcal{T}, d)$ that converges strongly to $\sigma_\infty \in \mathcal{T}$. Use σ_∞ as a root. If $\sigma_n \not\to \sigma_\infty$ weakly, then there would exist $\sigma > \sigma_\infty$ such that $\sigma_n \geq \sigma$ for infinitely many n. Thus $d(\sigma_n, \sigma_\infty) \geq d(\sigma, \sigma_\infty) > 0$ for these n, which contradicts that $\sigma_n \to \sigma_\infty$ strongly. □

Remark 6.7. If $\mathcal{T}$ is as in Example 6.5 with X infinite and $g(x) \equiv 1$, then the weak and strong topologies are not equivalent: if $(x_n)_{n\geq 0}$ are distinct elements in X then $(x_n, 1)$ converges weakly, but not strongly to $(x, 0)$.

6.4. **Trees associated to ultrametric spaces.** We next show how to construct a tree from an ultrametric space. This will be used to illustrate the connection between toroidal valuations and curves in Section 15. Recall that a metric d on a space X is an *ultrametric* if $d(x, y) \leq \max\{d(x, z), d(y, z)\}$ for any $x, y, z \in X$.

Let X be an ultrametric space of diameter 1. Define an equivalence relation $\sim$ on $X \times (0, 1)$ by declaring $(x, s) \sim (y, t)$ iff $d(x, y) \leq s = t$. Note that $(x, 1) \sim (y, 1)$ for any x, y. The set $\mathcal{T}_X$ of equivalence classes is a tree rooted at $(x, 1)$ under the partial order $(x, s) \leq (y, t)$ iff $d(x, y) \leq s \geq t$. It is a metric tree under the metric defined by $d((x, s), (x, t)) = |s - t|$ and $d(\sigma, \tau) = d(\sigma, \sigma \wedge \tau) + d(\tau, \sigma \wedge \tau)$ for general $\sigma, \tau \in \mathcal{T}_X$.

If the diameter of any ball of radius r equals r (!), then we may think of $(x, t) \in \mathcal{T}_X$ as the closed ball of radius t in X centered at x.

7. Tree structure on $\mathcal{V}$

We now show that the natural partial ordering on valuation space $\mathcal{V}$ turns it into a (non-metric) tree, the *valuative tree*, and describe its structure.

7.1. **Tree structure.** Recall the partial order $\leq$ defined on $\mathcal{V}$: $\nu \leq \mu$ iff $\nu(\phi) \leq \mu(\phi)$ for all $\phi \in R$.

Theorem 7.1. *Valuation space $\mathcal{V}$ is a complete tree rooted at $\nu_{\mathfrak{m}}$.*

The general properties of trees then give:

Corollary 7.2. *Any subset of $\mathcal{V}$ admits an infimum.*

Remark 7.3. If (ν_i) are valuations in $\mathcal{V}$, then $\nu := \wedge \nu_i \in \mathcal{V}$ can be constructed using the properties $\nu(\phi) = \inf_i \nu_i(\phi)$ for all irreducible $\phi \in \mathfrak{m}$ and $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ for all $\phi, \psi \in \mathfrak{m}$.

This construction does not work on more general rings. For instance, consider monomial valuations ν_i, $i = 1, 2, 3$ on $\mathbf{C}[[x_1, x_2, x_3]]$ defined by $\nu_i(x_j) = 3$ if $i \neq j$ and $\nu_i(x_i) = 1$. Define $\nu = \min_i \nu_i$ by the construction above and consider $\phi = x_1x_2 - x_2x_3 + x_3x_1$, $\psi = x_1x_2 + x_2x_3 - x_3x_1$. Then $\nu(\phi) = \nu(\psi) = 4$ but $\nu(\phi - \psi) = 2$, so ν is not a valuation.

A similar calculation shows that the natural partial order on the set of centered valuations on $\mathbf{C}[[x_1, x_2, x_3]]$ does *not* define a tree structure.

Recall that an automorphism $f : R \to R$ induces a bijection $f_* : \mathcal{V} \to \mathcal{V}$ given by $f_*(\nu)(\phi) = \nu(f(\phi))$. If $\mu \leq \nu$, then $f_*\mu \leq f_*\nu$. Hence we get:

Proposition 7.4. *Any ring automorphism $f : R \to R$ induces a tree isomorphism $f_* : \mathcal{V} \to \mathcal{V}$ with $f_*\nu_\mathfrak{m} = \nu_\mathfrak{m}$.*

Theorem 7.1 is proved by describing the partial order on $\mathcal{V}$ in terms of STKP's:

Proposition 7.5. *Consider $\nu, \nu' \in \mathcal{V}$ with $1 = \nu(x) = \nu'(x) \leq \min\{\nu(y), \nu'(y)\}$. Write $\nu = \mathrm{val}\,[(U_j); (\tilde{\beta}_j)]$, $\nu' = \mathrm{val}\,[(U'_j); (\tilde{\beta}'_j)]$. Then $\nu < \nu'$ iff $\nu \neq \nu'$ and*

$$\mathrm{length}\,(\nu') \geq \mathrm{length}\,(\nu) =: k < \infty, \quad U_j = U'_j \textit{ for } 0 \leq j \leq k \quad \textit{and} \quad \tilde{\beta}'_k \geq \tilde{\beta}_k.$$

Proof. First suppose the three displayed conditions hold. Then property (Q2) of Theorem 2.7 implies $\nu' \geq \nu_0 = \mathrm{val}\,[(U_j)_0^k; (\tilde{\beta}_j)_0^{k-1}, \tilde{\beta}'_k]$. But $\nu_0 \geq \nu$, so $\nu' \geq \nu$.

Conversely assume $\mathrm{val}\,[(U'_j); (\tilde{\beta}'_j)] = \nu' > \nu = \mathrm{val}\,[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ with $1 \leq k \leq \infty$. Let us show inductively that $U'_j = U_j$ for $j \leq k$. This is true by definition for $j = 0, 1$. Assume we proved it for $j < k$. Define $\nu_{j-1} = \mathrm{val}\,[(U_l)_0^{j-1}; (\tilde{\beta}_l)_0^{j-1}]$. Then $\nu'(U_j) \geq \nu(U_j) > \nu_{j-1}(U_j)$. Hence Theorem 4.1 implies $U'_j = U_j$. Finally, $k < \infty$ (or else $\nu = \nu'$) and $\tilde{\beta}'_k = \nu'(U_k) \geq \nu(U_k) = \tilde{\beta}_k$. □

Proof of Theorem 7.1. Fix $\nu \in \mathcal{V}$ with $\nu > \nu_\mathfrak{m}$. We will show that the set $I = \{\mu \mid \nu_\mathfrak{m} \leq \mu \leq \nu\}$ is a totally ordered set isomorphic to an interval in $\overline{\mathbf{R}}_+$.

Write $\nu = \mathrm{val}\,[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$. First assume $1 = \nu(x) \leq \nu(y)$ and $k < \infty$. Set $d_j = \deg_y(U_j)$ for $j \leq k$ ($d_0 = \infty$) and recall that the sequence $(\tilde{\beta}_j/d_j)$ is strictly increasing. We claim that I is isomorphic to the interval $J = [1, \tilde{\beta}_k/d_k]$. To see this, pick $t \in J$. There exists a unique integer $l \in [1, k]$ such that $\tilde{\beta}_{l-1}/d_{l-1} < t \leq \tilde{\beta}_l/d_l$. Set $\nu_t = \mathrm{val}\,[(U_j)_0^l; (\tilde{\beta}_j)_{j=0}^{l-1}, td_l]$. Proposition 7.5 then shows that this gives an isomorphism from J onto I. The case $k = \infty$ is treated in a similar way.

The remaining case to check is when $1 = \nu(y) < \nu(x)$. But then we may apply the automorphism f of R exchanging x and y and use Proposition 7.4. The details are left to the reader. Hence $\mathcal{V}$ is a tree.

The tree $\mathcal{V}$ is rooted at $\nu_\mathfrak{m}$ as $\nu_\mathfrak{m} \leq \nu$ for any valuation ν. To show completeness, pick an increasing sequence (ν_k) in $\mathcal{V}$. Let us assume $1 = \nu_k(x) \leq \nu_k(y)$ for all k. By Proposition 7.5 the STKP defining ν_k has increasing length. When $\mathrm{length}\,\nu_k \to \infty$ Theorem 2.21 shows that ν_k tends to a curve valuation or an infinitely singular valuation dominating all the ν_k's. Otherwise $\mathrm{length}\,(\nu_k)$ is constant for k large and we can write $\nu_k = \mathrm{val}\,[(U_i)_0^n; (\tilde{\beta}_i)_0^{n-1}, \tilde{\beta}_n^k]$. By Proposition 7.5, the sequence $\tilde{\beta}_n^k$ increases, hence converges to $\tilde{\beta}_n \in \overline{\mathbf{R}}_+$. The valuation $\nu = \mathrm{val}\,[(U_i)_0^n; (\tilde{\beta}_i)_0^n]$ dominates all the ν_k. Thus $\mathcal{V}$ is a complete tree. □

7.2. **Dendrology.** We now turn to the fine tree structure of valuation space $\mathcal{V}$.

Proposition 7.6. *The tree structure on valuation space $\mathcal{V}$ satisfies:*

(a) *the root of $\mathcal{V}$ is the $\mathfrak{m}$-adic valuation $\nu_\mathfrak{m}$;*
(b) *the ends of $\mathcal{V}$ are the infinitely singular and curve valuations;*
(c) *any tangent vector in $\mathcal{V}$ is represented by a curve valuation;*
(d) *the regular points of $\mathcal{V}$ are the irrational valuations;*
(e) *the branch points of $\mathcal{V}$ are the divisorial valuations. Further, the tangent space at a divisorial valuation is in bijection with $\mathbf{P}^1$.*

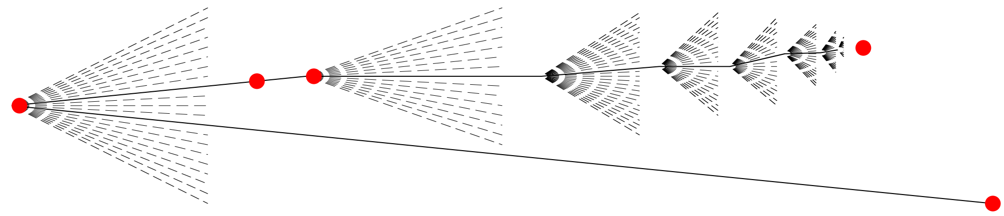

FIGURE 1. Valuation space $\mathcal{V}$. The valuations marked by dots are, from left to right: the $\mathfrak{m}$-adic valuation $\nu_{\mathfrak{m}}$, an irrational valuation, a divisorial valuation, an infinitely singular valuation and a curve valuation. The horizontal position indicates skewness (so the infinitely singular valuation has finite skewness in this case).

We will have more to say about (e) in Section 11.3.

Proof. We have already proved (a). As for (b), any STKP of length $k < \infty$ with $\tilde{\beta}_k < \infty$ can be extended to an STKP of length $k+1$. By Proposition 7.5 this means that no toroidal valuation is an end. The same proposition also shows that no valuation with an STKP of infinite length or with length $k < \infty$ and $\tilde{\beta}_k = \infty$ can be dominated by another valuation. Thus curve valuations and infinitely singular valuations are ends in $\mathcal{V}$, proving (b). A similar argument proves (c).

Next consider a toroidal valuation ν and $\vec{v} \in T_\nu\mathcal{V}$ represented by a curve valuation ν_ϕ. First assume that $\nu_\phi \wedge \nu < \nu$. Then the segment $]\nu, \nu_\phi]$ intersects $]\nu, \nu_{\mathfrak{m}}]$ at $\nu_\phi \wedge \nu$ so $\vec{v}$ is the tangent vector represented by $\nu_{\mathfrak{m}}$.

Otherwise, $\nu_\phi > \nu$. If ν is irrational and $\nu_\psi \geq \nu$ is another curve valuation, then $\mu := \nu_\phi \wedge \nu_\psi$ is toroidal by (b). But $\mu(R) \subset \mathbf{Q}$, so $\operatorname{rat.rk}(\mu) = 1$. Thus μ is divisorial, so $\mu > \nu$. It follows that $]\nu, \nu_\psi]$ and $]\nu, \nu_\phi]$ intersect. This proves (d).

When ν is divisorial, write $\nu = \operatorname{val}[(U_j)_0^k, (\tilde{\beta}_j)_0^k]$. For $\mu > \nu$, either $\mu(U_k) > \nu(U_k) = \tilde{\beta}_k$, or $\mu(U_k) = \tilde{\beta}_k$. In the former case, μ represents the same vector as $\nu_\infty = \operatorname{val}[(U_j)_0^k, (\tilde{\beta}_j)_0^{k-1}, \infty]$. In the latter case Theorem 2.7 (A2) shows that there exists a polynomial $U_{k+1} = U_k^{n_k} - \theta \prod_0^{k-1} U_j^{m_{kj}}$ with $\theta = \theta(\mu) \in \mathbf{C}^*$, such that if $\phi \in R$ then $\mu(\phi) > \nu(\phi)$ iff U_{k+1} divides ϕ in $\operatorname{gr}_\mu \mathbf{C}(x)[y]$. Define $\nu_\theta = \nu_{U_{k+1}}$ and $\tilde{\beta}_{k+1} = \mu(U_{k+1})$. Then $\mu \wedge \nu_\theta \geq \operatorname{val}[(U_j)_0^{k+1}, (\tilde{\beta}_j)_0^{k+1}] > \nu$ so that μ and ν_θ define the same tangent vector. Conversely $\nu_\theta \wedge \nu_{\theta'} = \nu$ as soon as $\theta \neq \theta'$. The set of tangent vectors at ν is hence in bijection with $\{\nu_{\mathfrak{m}}, \nu_\infty, \nu_\theta\}_{\theta \in \mathbf{C}^*}$ which is in bijection with $\mathbf{P}^1$. □

Remark 7.7. The (non-metric) tree structure on $\mathcal{V}$ does not allow us to distinguish between a curve valuation and an infinitely singular valuation. In Section 16 we will define the *multiplicity* of a valuation. This multiplicity is an increasing function on $\mathcal{V}$ with values in $\mathbf{N} \cup \{\infty\}$ and the infinitely singular valuations are the ones with infinite multiplicity.

Remark 7.8. The identification of valuations with STKP's gives an explicit model for the tree structure on $\mathcal{V}$. Namely, let $\mathcal{T}$ be the set consisting of (0) and of pairs $(\bar{s}, \bar{\theta})$ with $\bar{s} = (s_1, \dots, s_{m+1})$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, $0 \leq m \leq \infty$, $s_i \in \mathbf{Q}_+^*$ for

$1 \le i < m+1$, $s_{m+1} \in (0,\infty]$ if $m < \infty$, $\theta_i \in \mathbf{C}^*$. Define a partial ordering on $\mathcal{T}$ by $(\bar{s},\bar{\theta}) \le (\bar{s}',\bar{\theta}')$ iff $m \le m'$, $s_i = s'_i$ and $\theta_i = \theta'_i$ for $i < m$ and $s_m \le s'_m$ if $m > 1$, and if $m = 1$, $1 \le s_1 \le s'_1$ or $s'_1 \le s_1 \le 1$.

Define $\imath : \mathcal{T} \to \mathcal{V}$ as follows. First set $\imath(0) = \nu_x$. If $\sigma = (\bar{s},\bar{\theta}) \in \mathcal{T}$, then set $\tilde{\beta}_0 = 1/\min\{1, s_1\}$, $\tilde{\beta}_1 = s_1/\min\{1, s_1\}$, and define, inductively, $n_k = \min\{n \mid n\tilde{\beta}_k \in \sum_0^{k-1} \tilde{\beta}_i \mathbf{Z}\}$ and $\tilde{\beta}_{k+1} := n_k\tilde{\beta}_k + s_{k+1}$. Define U_k by (P2) and set $\imath(\sigma) := \operatorname{val}[(U_k);(\tilde{\beta}_k)]$. Proposition 7.5 shows that $\imath$ is a tree isomorphism.

8. Metric tree structure on $\mathcal{V}$

Our next goal is to equip the valuative tree $\mathcal{V}$ with a natural tree metric.

8.1. **Skewness.** We first introduce a new invariant of a valuation which in an intrinsic way measures how far the valuation is from the $\mathfrak{m}$-adic valuation $\nu_{\mathfrak{m}}$.

Definition 8.1. For $\nu \in \mathcal{V}$, define the *skewness* $\alpha(\nu) \in [1,\infty]$ by

$$\alpha(\nu) := \sup\left\{ \frac{\nu(\phi)}{m(\phi)} \;\middle|\; \phi \in \mathfrak{m} \right\}. \tag{8.1}$$

This quantity is an invariant of a valuation, in the following sense:

Proposition 8.2. *If $f : R \to R$ is a ring automorphism and $f_* : \mathcal{V} \to \mathcal{V}$ is the induced map on valuation space, then $\alpha(f_*\nu) = \alpha(\nu)$ for any $\nu \in \mathcal{V}$.*

Proof. This is immediate since

$$\alpha(f_*\nu) = \sup_{\phi\in\mathfrak{m}} \frac{\nu(f(\phi))}{m(f(\phi))} = \sup_{\psi\in\mathfrak{m}} \frac{\nu(\psi)}{m(\psi)} = \alpha(\nu).$$

□

Let us compute the skewness of a valuation in terms of its STKP.

Proposition 8.3. *Let $\nu \in \mathcal{V}$ and write $\nu = \operatorname{val}[(U_i)_0^k;(\tilde{\beta}_i)_0^k]$.*

(i) *If ν is toroidal, then $\alpha(\nu) = d_k^{-1}\tilde{\beta}_k\tilde{\beta}_0$, where $d_k = \deg_y U_k$. In particular $\alpha(\nu)$ is rational iff ν is divisorial.*
(ii) *If ν is a curve valuation, then $\alpha(\nu) = \infty$.*
(iii) *If ν is infinitely singular, then $\alpha(\nu) = \lim_{j\to\infty} d_j^{-1}\tilde{\beta}_j\tilde{\beta}_0$, which can be any number in $(1,\infty]$.*

Proof. If $\nu = \nu_\phi$ is a curve valuation, then $\alpha(\nu) \ge \nu(\phi)/m(\phi) = \infty$, which proves (ii). As for (i) and (iii) note that it suffices to use ϕ irreducible in (8.1). Indeed, if $\nu(\phi) \le \alpha m(\phi)$ for $\phi \in \mathfrak{m}$ irreducible and $\phi = \prod \phi_i$ is reducible, then

$$\nu(\phi) = \sum_i \nu(\phi_i) \le \alpha \sum_i m(\phi_i) = \alpha m(\phi).$$

So assume ϕ irreducible and write $\nu_\phi = \operatorname{val}[(U_j^\phi);(\tilde{\beta}_j^\phi)]$. Let $l = \operatorname{con}(\nu,\nu_\phi)$ as defined in (5.1). Then $l \le k$. Recall that the sequence $(d_j^{-1}\tilde{\beta}_j)_1^k$ is increasing. We apply Proposition 5.2:

$$\frac{\nu(\phi)}{m(\phi)} = d_l^{-1}\min\{\tilde{\beta}_l,\tilde{\beta}_l^\phi\}\min\{\tilde{\beta}_0,\tilde{\beta}_0^\phi\} \le \sup_{j\le k} d_j^{-1}\tilde{\beta}_j\tilde{\beta}_0. \tag{8.2}$$

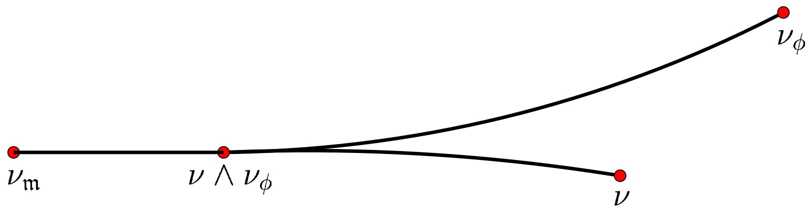

FIGURE 2. The value $\nu(\phi)$ for $\phi \in \mathfrak{m}$ irreducible depends on the multiplicity $m(\phi)$ and the relative position of ν and the curve valuation ν_ϕ in the valuative tree $\mathcal{V}$. See Proposition 8.4.

If ν is toroidal, i.e. $k < \infty$, then equality holds in (8.2) if $l = k$ and $\tilde{\beta}^\phi_k = \tilde{\beta}_k$, proving (i).

If ν is infinitely singular, so that $k < \infty$, then for any j we may pick ϕ with $l = j$ and $\tilde{\beta}^\phi_l = \tilde{\beta}_l$. Then $\nu(\phi)/m(\phi) \geq d_j^{-1}\tilde{\beta}_j\tilde{\beta}_0$. Letting $j \to \infty$ yields (iii). □

As a first example of how skewness can be used, let us prove the following result, which will be used repeatedly in the sequel.

Proposition 8.4. *For any valuation $\nu \in \mathcal{V}$ and any irreducible $\phi \in \mathfrak{m}$ we have*

$$\nu(\phi) = \alpha(\nu \wedge \nu_\phi) m(\phi).$$

In particular $\nu(\phi) \leq \alpha(\nu) m(\phi)$ with equality iff $\nu_\phi \geq \nu$.

Proof. Keep the notation of Proposition 8.3 and its proof. If $\nu = \nu_\phi$, then $\nu \wedge \nu_\phi = \nu_\phi$ and the result follows from (ii). Otherwise $l := \mathrm{con}\,(\nu_\phi, \nu) < \infty$ and $\nu \wedge \nu_\phi = \mathrm{val}\,[(U_j)_0^l; (\tilde{\beta}_j)_0^{l-1}, \tilde{\beta}]$, where $\tilde{\beta} = \min(\tilde{\beta}_l, \tilde{\beta}^\phi_l)$. Then

$$\alpha(\nu \wedge \nu_\phi) = d_l^{-1}\tilde{\beta}\tilde{\beta}_0 = \nu(\phi)/m(\phi),$$

which completes the proof. □

Remark 8.5. Skewness is closely related to *volume* as defined in [ELS]. Indeed, if we define $\mathrm{Vol}\,(\nu) = \limsup_{c\to\infty} 2c^{-2} \dim_{\mathbf{C}} R/\{\nu \geq c\}$, then we have

$$\mathrm{Vol}\,(\nu) = \alpha(\nu)^{-1}. \tag{8.3}$$

We sketch the proof, referring to [ELS, Example 3.15] for details. Write $\nu = \mathrm{val}\,[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ and suppose for simplicity that k is finite. For any $c > 0$, a basis for the vector space $R/\{\nu \geq c\}$ is given by the monomials $U_0^{i_0} \dots U_k^{i_k}$ where $0 \leq i_j < n_j$ for $1 \leq j \leq k-1$, and $\sum_0^k i_j \tilde{\beta}_j < c$. The number of such monomials is given (up to a bounded function) by $\prod_1^{k-1} n_j \cdot \mathrm{Area}\{i_0 + \tilde{\beta}_k i_k \leq c \mid i_0, i_k \geq 0\} \simeq d_k c^2 / 2\tilde{\beta}_k$. This gives (8.3).

8.2. **Parameterizations.** We now show that skewness naturally parameterizes valuation space:

Proposition-Definition 8.6. *Fix $\phi \in \mathfrak{m}$ irreducible and pick $t \in [1, \infty]$. Then there is a unique valuation $\nu = \nu_{\phi,t}$ in the segment $[\nu_{\mathfrak{m}}, \nu_\phi]$ with skewness $\alpha(\nu) = t$. Further, if $\nu \in \mathcal{V}$ satisfies $\nu(\phi) \geq t\, m(\phi)$, then $\nu \geq \nu_{\phi,t}$.*

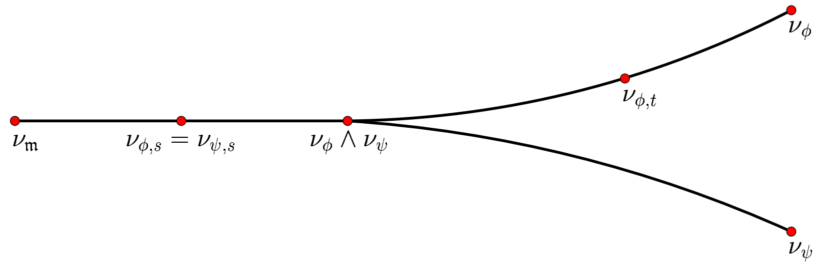

FIGURE 3. Skewness is used to parameterize the segment $[\nu_\mathfrak{m}, \nu_\phi]$ by $\nu_{\phi,t}$, $1 \leq t \leq \infty$. We have $\nu_{\phi,t} = \nu_{\psi,s}$ iff $s = t \leq \alpha(\nu_\phi \wedge \nu_\psi)$. See Proposition 8.6.

In other words, the function $\alpha : \mathcal{V} \to [1, \infty]$ defines a parameterization. The latter is essentially equivalent to the parameterization by STKP's but has the appeal of being simpler and coordinate independent.

Proof of Proposition 8.6. To construct $\nu_{\phi,t}$ we consider the curve valuation ν_ϕ and write $\nu_\phi = \text{val}\,[(U_i); (\tilde{\beta}_i)]$. We can assume that $\tilde{\beta}_0 = 1 \leq \tilde{\beta}_1$ by permuting coordinates if necessary. Pick $t \geq 1$, and define $\nu_{\phi,t}$ as follows: if $t = 1, \infty$, then $\nu = \nu_\mathfrak{m}$ and $\nu = \nu_\phi$, respectively. If $1 < t < \infty$, then there exists a unique $k \geq 1$ with $t \in]\tilde{\beta}_{k-1}/d_{k-1}, \tilde{\beta}_k/d_k]$, where $d_k = \deg_y U_k$ (and $d_0 = 1$). We set

$$\nu_{\phi,t} := \text{val}\,[(U_i)_0^k; (\tilde{\beta}_i)_0^{k-1}, t \cdot d_k].$$

Then $\nu_{\phi,t} \in [\nu_\mathfrak{m}, \nu_\phi]$ and by Propositions 8.3 and 8.4 we have $\nu_{\phi,t}(\phi) = tm(\phi)$ and $\alpha(\nu) = t$.

On the other hand, if $\nu \in \mathcal{V}$ satisfies $\nu(\phi) = tm(\phi)$, then $\alpha(\nu \wedge \nu_\phi) = t$ by Proposition 8.4. Since $\nu \wedge \nu_\phi \in [\nu_\mathfrak{m}, \nu_\phi]$ the previous argument implies $\nu \wedge \nu_\phi = \nu_{\phi,t}$ and hence $\nu \geq \nu_{\phi,t}$. □

Corollary 8.7. *If $\nu \in \mathcal{V}$, $\phi \in \mathfrak{m}$ is irreducible and $\nu(\phi)$ is irrational, then $\nu = \nu_{\phi,t}$ where $t = \nu(\phi)/m(\phi)$.*

Proof. Define $t = \nu(\phi)/m(\phi)$. By Proposition 8.6 we have $\nu \geq \nu_{\phi,t}$. Since $\nu_{\phi,t}$ is irrational, either equality holds, or else $\nu \geq \nu_{\phi,s}$ for some $s > t$, in which case $\nu(\phi) \geq sm(\phi) > tm(\phi)$, a contradiction. □

8.3. **An invariant tree metric.** For $\nu, \mu \in \mathcal{V}$ set

$$d_\mathcal{V}(\mu, \nu) = \left(\frac{1}{\alpha(\mu \wedge \nu)} - \frac{1}{\alpha(\mu)}\right) + \left(\frac{1}{\alpha(\mu \wedge \nu)} - \frac{1}{\alpha(\nu)}\right). \tag{8.4}$$

Theorem 8.8. *The metric $d_\mathcal{V}$ gives valuation space $\mathcal{V}$ the structure of a metric tree. Further, $d_\mathcal{V}$ is complete and is an invariant metric in the sense that if $f : R \to R$ is an automorphism and $f_* : \mathcal{V} \to \mathcal{V}$ is the induced map on valuation space, then $d_\mathcal{V}(f_*\nu, f_*\mu) = d_\mathcal{V}(\nu, \mu)$ for any $\nu, \mu \in \mathcal{V}$.*

Proof. We know by Proposition 8.6 that skewness $\alpha : \mathcal{V} \to [1, \infty]$ defines a parameterization of $\mathcal{V}$. A simple adaptation of the proof of Lemma 6.4 then shows that $(\mathcal{V}, d_{\mathcal{V}})$ is a metric tree. Completeness of $(\mathcal{V}, d_{\mathcal{V}})$ follows from completeness of $\mathcal{V}$ as a tree. Invariance follows from Proposition 8.2. □

Part 4. Topologies

In this part we analyze different topologies on $\mathcal{V}$. We start in Section 9 by considering the weak topology, defined either in terms of pointwise convergence or in terms of the non-metric tree structure. Then in Section 10 we study the strong topology, induced either by uniform convergence or by a natural tree metric. Next we have the Zariski topology on the set of all centered Krull valuations. This is analyzed in Section 11. A refinement of the Zariski topology is the Hausdorff-Zariski topology, considered in Section 12. Finally we compare the different topologies on $\mathcal{V}$ in Section 13.

9. The weak topology

At this stage we have two weak topologies on valuation space: the weak topology defined by pointwise convergence and the weak tree topology induced by the tree structure on $\mathcal{V}$. In this section we show that these two topologies are in fact identical. We also study its main properties.

9.1. The equivalence.

We defined the weak topology on $\mathcal{V}$ by describing converging sequences: $\nu_k \rightharpoonup \nu$ iff $\nu_k(\phi) \to \nu(\phi)$ for every $\phi \in \mathfrak{m}$. Equivalently, a basis of open sets is given by $\{\nu \in \mathcal{V} \mid t < \nu(\phi) < t'\}$ over $\phi \in \mathfrak{m}$ irreducible and $t' > t \geq 1$. The weak tree topology is generated by $\{\mathcal{V}_{\vec{v}}\}$, where $\vec{v}$ runs over all tangent vectors in $\mathcal{V}$.

Theorem 9.1. *The weak topology on $\mathcal{V}$ coincides with the weak tree topology.*

Proof. We use Proposition 8.4 repeatedly. If $\vec{v} \in T_\nu\mathcal{V}$ is a tangent vector not represented by $\nu_\mathfrak{m}$ (true e.g. if $\nu = \nu_\mathfrak{m}$), then $\mathcal{V}_{\vec{v}} = \{\mu \mid \mu(\phi) > \alpha\, m(\phi)\}$, where $\alpha = \alpha(\nu)$ and ν_ϕ represents $\vec{v}$. If instead $\vec{v}$ is represented by $\nu_\mathfrak{m}$, then $V_{\vec{v}} = \{\mu \in \mathcal{V} \mid \mu(\phi) < \alpha\, m(\phi)\}$, where $\nu_\phi \geq \nu$. In both cases $\mathcal{V}_{\vec{v}}$ is open in the weak topology.

Conversely, every nontrivial set of the form $\{\nu(\phi) > t\}$ or $\{\nu(\phi) < t\}$ with $t \geq 1$ and $\phi \in \mathfrak{m}$ irreducible is of the form $\mathcal{V}_{\vec{v}}$. This completes the proof. □

9.2. Properties.

We now investigate the weak topology further.

Proposition 9.2. *The weak topology is compact but not metrizable.*

Proof. Compactness follows from Theorem 9.1 and the fact that any complete tree is weakly compact (see Proposition 6.3). Alternatively, $\mathcal{V}$ is naturally embedded as a closed subspace of the compact product space $[0, \infty]^R$.

To prove that $\mathcal{V}$ is not metrizable it suffices to show that $\nu_\mathfrak{m} \in \mathcal{V}$ has no countable basis of open neighborhoods. For $\theta \in \mathbf{C}^*$, let $\nu_\theta := \nu_{y+\theta x,2}$ (see Proposition 8.6) and $\Omega_\theta := \{\nu \not\geq \nu_\theta\}$. Any Ω_θ is an open neighborhood of $\nu_\mathfrak{m}$. Suppose $\{V_k\}_{k\geq 1}$ is a countable basis of open neighborhoods of $\nu_\mathfrak{m}$. Then for any θ there exists $k = k(\theta) \geq 0$ with $V_k \subset \Omega_\theta$. As $\mathbf{C}^*$ is uncountable, one can find k and a sequence θ_i with $\theta_i \neq \theta_j$ for $i \neq j$ such that $V_k \subset \Omega_{\theta_i}$. Now $\nu_{\theta_i} \rightharpoonup \nu_\mathfrak{m}$, so $\nu_{\theta_i} \in V_k$ for $i \gg 1$. But $\nu_{\theta_i} \notin \Omega_{\theta_i}$. This is a contradiction. □

Proposition 9.3. *The four subsets of $\mathcal{V}$ consisting of divisorial, irrational, infinitely singular and curve valuations are all weakly dense in $\mathcal{V}$.*

Proof. Instead of proving all of this directly we will appeal to Proposition 10.3 which asserts that the divisorial, irrational and infinitely singular valuations are all (individually) dense in $\mathcal{V}$ in a stronger topology that the weak topology. Hence it suffices to show that, say, any divisorial valuation can be weakly approximated by curve valuations. But if ν is divisorial, then by Proposition 7.6 there exists a sequence ϕ_n of irreducible elements in $\mathfrak{m}$ such that the associated curve valuations ν_{ϕ_n} represent distinct tangent vectors at ν. Then $\nu_{\phi_n} \rightharpoonup \nu$ as $n \to \infty$. □

10. The strong topology

As with the weak topology, we have two candidates for the strong topology on $\mathcal{V}$: one defined using uniform sequential convergence (see (10.1) below) and one induced by the tree metric (8.4) on $\mathcal{V}$. We now show that these two coincide and analyze the properties of the strong topology.

10.1. **The equivalence.** A strong topology on $\mathcal{V}$ can be defined in a quite general setting (i.e. for other rings than R) in terms of the metric

$$d_{\mathrm{str}}(\nu_1, \nu_2) = \sup_{\phi \in \mathfrak{m} \text{ irreducible}} \left| \frac{m(\phi)}{\nu_1(\phi)} - \frac{m(\phi)}{\nu_2(\phi)} \right| \tag{10.1}$$

This topology is stronger than the weak topology and any automorphism $f : R \to R$ induces an isometry $f_* : \mathcal{V} \to \mathcal{V}$ for d_{str}. In fact, the metric d_{str} is compatible with the tree metric $d_{\mathcal{V}}$ defined in (8.4):

Theorem 10.1. *The strong topology on $\mathcal{V}$ is identical to the strong tree topology. More precisely, if $d_{\mathcal{V}}$ is the tree metric on $\mathcal{V}$ given by* (8.4)*, then, for $\nu_1, \nu_2 \in \mathcal{V}$:*

$$d_{\mathrm{str}}(\nu_1, \nu_2) \le d_{\mathcal{V}}(\nu_1, \nu_2) \le 2 d_{\mathrm{str}}(\nu_1, \nu_2) \tag{10.2}$$

Proof. Let us consider $\phi \in \mathfrak{m}$ irreducible. Set $\nu = \nu_1 \wedge \nu_2$. We first show that

$$\left| \nu_1^{-1}(\phi) - \nu_2^{-1}(\phi) \right| = \max_{i=1,2} \left\{ \nu^{-1}(\phi) - \nu_i^{-1}(\phi) \right\}. \tag{10.3}$$

To see this, notice that Proposition 8.4 implies $\mu(\phi) = (\mu \wedge \nu_\phi)(\phi)$ for any $\mu \in \mathcal{V}$. Thus we may replace ν_i by $\nu_i \wedge \nu_\phi$, so that $\nu \le \nu_i \le \nu_\phi$, $i = 1, 2$. But this means that $\nu = \nu_1 \le \nu_2 \le \nu_\phi$ or $\nu = \nu_2 \le \nu_1 \le \nu_\phi$ and then (10.3) is immediate.

Multiplying (10.3) by $m(\phi)$ and taking the supremum over $\phi \in \mathfrak{m}$ we get $d_{\mathrm{str}}(\nu_1, \nu_2) = \max_i d_{\mathcal{V}}(\nu_i, \nu)$, which implies (10.2). □

10.2. **Properties.** We now investigate the strong topology further.

Proposition 10.2. *The strong topology on $\mathcal{V}$ is strictly stronger than the weak topology. It is not locally compact.*

Proof. The last assertion implies the first one as the weak topology is compact. Consider $\nu \in \mathcal{V}$ divisorial and pick $\phi_n \in \mathfrak{m}$ irreducible with $\nu_{\phi_n} > \nu$, such that ν_{ϕ_n} represent distinct tangent vectors at ν. For fixed $\varepsilon > 0$, set $\nu_n = \nu_{\phi_n, \alpha+\varepsilon}$, where α is the skewness of ν. Any ν_n is at distance $\varepsilon/\alpha(\alpha+\varepsilon)$ from ν. Further, $\nu_n \rightharpoonup \nu$, so ν_n has no strong accumulation point. If ν had a compact strong neighborhood, it would contain a ball of positive radius, say bigger than ε. Then ν_n would have a strongly convergent subsequence. This is impossible. □

Proposition 10.3. *The three subsets of $\mathcal{V}$ consisting of divisorial, irrational and infinitely singular valuations are all strongly dense in $\mathcal{V}$.*

The strong closure of the set $\mathcal{C}$ of curve valuations is the set of valuations of infinite skewness. In particular, $\overline{\mathcal{C}}\setminus\mathcal{C}$ contains only infinitely singular valuations.

Proof. For the first assertion, we first prove that any valuation $\nu \in \mathcal{V}$ can be strongly approximated by divisorial and irrational valuations. If ν is not infinitely singular, then $\nu = \nu_{\phi,t}$ for some irreducible $\phi \in \mathfrak{m}$ and $t \in [1, \infty]$. Then $\mu := \nu_{\phi,s}$ converges strongly to ν as $s \to t$ and μ is divisorial (irrational) if s is rational (irrational). If ν is infinitely singular, then $\nu = \lim \nu_n$ for an increasing sequence ν_n, where ν_n can be chosen to be all divisorial or all irrational.

To complete the proof of the first assertion we show that any divisorial valuation can be approximated by an infinitely singular valuation. So let $\nu = \operatorname{val}[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ be divisorial and fix $\varepsilon > 0$. Pick an arbitrary $\theta_k \in \mathbf{C}^*$ and define $U_{k+1} := U_k^{n_k} - \theta_k \prod_0^{k-1} U_j^{m_{k,j}}$. where n_k and $m_{k,j}$ are defined by (2.2).

Pick $\tilde{\beta}_{k+1}$ rational with $0 < \tilde{\beta}_{k+1} - n_k\tilde{\beta}_k < 2^{-1}d_{k+1}\varepsilon$, so that $\tilde{\beta}_{k+1}$ does not belong to the discrete subgroup $\sum_0^k \mathbf{Z}\tilde{\beta}_j$ of $\mathbf{Q}$. Set $\nu_{k+1} := \operatorname{val}[(U_i)_0^{k+1}; (\tilde{\beta}_i)_0^{k+1}]$. By construction $n_{k+1} \geq 2$. This method produces divisorial valuations $\nu_{j+k} = \operatorname{val}[(U_i)_0^{k+j}; (\tilde{\beta}_i)_0^{k+j}]$, $j \geq 1$ satisfying $n_{k+j} \geq 2$ and $0 < \tilde{\beta}_{k+j+1} - n_{k+j}\tilde{\beta}_{k+j} < 2^{-(j+1)}d_{k+j+1}\varepsilon$. The sequence $(\nu_{k+j})_j$ increases towards an infinitely singular valuation ν_∞, and

$$d_{\mathcal{V}}(\nu_\infty, \nu) = \alpha^{-1}(\nu) - \alpha^{-1}(\nu_\infty) = \lim_{j\to\infty} \tilde{\beta}_j^{-1}d_j - \tilde{\beta}_k^{-1}d_k < \sum_{j\geq 1} 2^{-j}\varepsilon = \varepsilon.$$

This concludes the proof of the first assertion.

For the second assertion, first notice that skewness defines a strongly continuous function $\alpha : \mathcal{V} \to [1, \infty]$, so since every curve valuation has infinite skewness, so does every valuation in $\overline{\mathcal{C}}$. Conversely if $\nu = \operatorname{val}[(U_i); (\tilde{\beta}_i)]$ has infinite skewness, then $\nu_n := \operatorname{val}[(U_i)_0^n; (\tilde{\beta}_i)_0^{n-1}, \infty]$ is a curve valuation with

$$d_{\mathcal{V}}(\nu_n, \nu) = \big(\alpha^{-1}(\nu \wedge \nu_n) - \alpha^{-1}(\nu)\big) + \big(\alpha^{-1}(\nu \wedge \nu_n) - \alpha^{-1}(\nu_n)\big) = 2\tilde{\beta}_n^{-1}d_n \to 0,$$

so $\nu_n \to \nu$ as $n \to \infty$. This completes the proof. □

We end this section by discussing another tree metric. By Proposition 10.3, the subset $\mathcal{V}'$ of $\mathcal{V}$ consisting of valuations with finite skewness is a strongly open subtree that contains all toroidal valuations. It is obviously not complete as a tree, but can be given a natural complete metric. Namely, for $\nu, \mu \in \mathcal{V}'$ set

$$d_{\mathcal{V}'}(\mu, \nu) = \sup_{\phi\in\mathfrak{m}} \left| \frac{\nu_1(\phi)}{m(\phi)} - \frac{\nu_2(\phi)}{m(\phi)} \right|$$

We may check that the supremum can be taken over irreducible ϕ's. Following the proof of Theorem 10.1, we infer that $d_{\mathcal{V}'}(\mu, \nu) = (\alpha(\mu) - \alpha(\mu \wedge \nu)) + (\alpha(\nu) - \alpha(\mu \wedge \nu))$.

Proposition 10.4. *The metric space $(\mathcal{V}', d_{\mathcal{V}'})$ is complete.*

Proof. Any $d_{\mathcal{V}'}$-Cauchy sequence ν_n in $\mathcal{V}'$ is $d_{\mathcal{V}}$-Cauchy, hence $d_{\mathcal{V}}(\nu_n, \nu) \to 0$ for some $\nu \in \mathcal{V}$ by completeness of $d_{\mathcal{V}}$. But it is easy to see that ν_n must have uniformly bounded skewness, so $\alpha(\nu) < \infty$ and then $d_{\mathcal{V}'}(\nu_n, \nu) \to 0$ as well. □

11. The Zariski topology.

We now turn to the Zariski topology, which is not Hausdorff as divisorial valuations do not define closed points. We show how to make it Hausdorff by identifying a divisorial valuation with the valuations in its closure. This gives back $\mathcal{V}$ endowed with the weak topology. We also show that tangent vectors in the tree $\mathcal{V}$ at a divisorial valuation are canonically associated with exceptional curve valuations.

11.1. **Definition.** The Zariski topology is defined on the set $\mathcal{V}_K$ of *all* centered Krull valuations on $R = \mathbf{C}[[x, y]]$ (not necessarily $\overline{\mathbf{R}}_+$-valued see Section 1.2). A Krull valuation ν is determined by its valuation ring $R_\nu = \{\nu \geq 0\} \subset K$, so an open set in $\mathcal{V}_K$ is a set of valuation rings satisfying certain conditions.

Definition 11.1. A basis for the Zariski topology on $\mathcal{V}_K$ is given by

$$V(A) := \{\nu \mid R_\nu \supset A\},$$

where A ranges over subrings of K of the form $A = R[z_1, \dots, z_k]$, $z_i \in K$.

Note that $V(A) \cap V(A') = V([A, A'])$, where $[A, A']$ denotes the algebra generated by A and A'. Thus we have a well-defined topology whose open sets are arbitrary union of $V(A)$'s.

Remark 11.2. The topological space $\mathcal{V}_K$ endowed with the Zariski topology is called the Riemann-Zariski variety of R (see [ZS], [V]).

Proposition 11.3. *$\mathcal{V}_K$ is quasi-compact but not Hausdorff. Moreover,*

(i) *Any non-divisorial valuation is a closed point.*
(ii) *The closure of a divisorial valuation associated to an exceptional curve E is the set of exceptional curve valuations $\nu_{E,p}$ whose centers lie on E.*

Proof. That $\mathcal{V}$ is not Hausdorff follows from (ii). Quasi-compactness will be proved in Theorem 12.2. Both (i) and (ii) are consequences of Lemma 1.4 since a valuation μ lies in the closure of another valuation $\nu \neq \mu$ iff $R_\mu \subsetneq R_\nu$. □

Remark 11.4. $\mathcal{V} \subset \mathcal{V}_K$ is neither open nor closed for the Zariski topology.

Remark 11.5. The Zariski topology can be geometrically described as follows: a basis is given by $V(\varpi, C)$, where ϖ ranges over compositions of finitely many blow-ups and C over Zariski open subsets of the exceptional divisor $\varpi^{-1}(\mathfrak{m})$. Here C is the complement in $\varpi^{-1}(\mathfrak{m})$ of finitely many irreducible components and finitely many points, and $V(\varpi, C)$ is the set of all $\nu \in \mathcal{V}_K$ whose associated sequence of infinitely nearby points $\Pi[\nu] = \{p_i\}$ are eventually in C.

11.2. **Recovering $\mathcal{V}$ from $\mathcal{V}_K$.** We can try to turn $\mathcal{V}_K$ into a Hausdorff space $\widetilde{\mathcal{V}}_K$ by identifying ν and ν' if they both belong to the Zariski closure of some valuation μ. By Proposition 11.3 this amounts to identifying exceptional curve valuations with their associated divisorial valuation. Let $p : \mathcal{V}_K \to \widetilde{\mathcal{V}}_K$ be the natural projection and endow $\widetilde{\mathcal{V}}_K$ with the quotient topology. Then $\widetilde{\mathcal{V}}_K$ is compact, being the image of a quasi-compact space by a continuous map.

Consider the natural injection $\imath : \mathcal{V} \to \mathcal{V}_K$, where $\mathcal{V}$ carries the weak topology.

Theorem 11.6. *The composition $p \circ \imath : \mathcal{V} \to \widetilde{\mathcal{V}}_K$ is a homeomorphism.*

Proof. Since $\mathcal{V}$ contains all divisorial valuations but no exceptional curve valuations, injectivity and surjectivity of $p \circ \imath$ follow from Proposition 11.3. Since $\mathcal{V}$ is Hausdorff and $\widetilde{\mathcal{V}_K}$ is compact, continuity of $(p \circ \imath)^{-1}$ will imply that $p \circ \imath$ is in fact a homeomorphism.

Therefore, let us show that $(p \circ \imath)^{-1}$ is continuous. A basis for the weak (tree) topology is given by the open sets $\mathcal{V}_{\vec{v}}$, over tangent vectors $\vec{v}$ at *divisorial* valuations $\nu \in \mathcal{V}$. For $\phi, \psi \in \mathfrak{m}$ irreducible and $t > 0$ define

$$V(\phi, \psi, t) := \left\{ \mu \in \mathcal{V} \;\middle|\; \frac{\mu(\phi)}{m(\phi)} < t \frac{\mu(\psi)}{m(\psi)} \right\}.$$

Proposition 8.4 implies that if $\vec{v} \in T_\nu \mathcal{V}$ is not represented by $\nu_{\mathfrak{m}}$ (true e.g. if $\nu = \nu_{\mathfrak{m}}$), then $\mathcal{V}_{\vec{v}} = V(\phi, \psi, 1)$, where ν_ψ represents $\vec{v}$ and $\nu_\phi \wedge \nu_\psi = \nu$. If instead $\vec{v}$ is represented by $\nu_{\mathfrak{m}}$, then $\mathcal{V}_{\vec{v}} = V(\phi, \psi, \alpha(\nu))$, where $\nu_\phi \geq \nu$ and $\nu_\psi \wedge \nu_\phi = \nu_{\mathfrak{m}}$.

Hence it suffices to show that $(p \circ \imath)(V)$ is open in $\widetilde{\mathcal{V}_K}$ for any $V = V(\phi, \psi, t)$. This amounts to $p^{-1}(p \circ \imath)(V)$ being open in $\mathcal{V}_K$. Define

$$W = W(\phi, \psi, t) := \bigcup_{p/q > t} \left\{ \nu \in \mathcal{V}_K \mid \nu\left(\psi^{pm(\phi)}/\phi^{qm(\psi)}\right) \geq 0 \right\}.$$

Then W is open in $\mathcal{V}_K$ and $\imath(V) = \imath(\mathcal{V}) \cap W$. We claim that if $\nu \in \mathcal{V}_K$ is divisorial and $\nu' \in \overline{\{\nu\}}$ an exceptional curve valuation, then $\nu \in W$ if and only if $\nu' \in W$. This claim easily implies that $p^{-1}(p \circ \imath)(V) = W$ is open, completing the proof.

As for the claim, $\nu \in W$ implies $t' := \nu(\psi)m(\phi)/\nu(\phi)m(\psi) > t$. Pick $t'' \in (t, t')$ rational, $t'' = p/q$. Then ν belongs to the closed set $\{\mu \mid \mu(\psi^{pm(\phi)}/\phi^{qm(\psi)}) > 0\} \subset W$, hence so does ν'. Conversely, if $\nu' \in W$ then $\nu \in W$ as $R_{\nu'} \subset R_\nu$. ☐

11.3. **The tangent space at a divisorial valuation.** Fix $\nu \in \mathcal{V}$ *divisorial.* We have seen in Proposition 7.6 that the tree tangent space $T_\nu \mathcal{V}$ is in bijection with $\mathbf{P}^1$. Here we make this isomorphism precise. As a byproduct we also see that tree tangent vectors act as a kind of derivative.

Theorem 11.7. *Let $\nu \in \mathcal{V}$ be a divisorial valuation. There is a natural 1-1 correspondence between tree tangent vectors $\vec{v}$ at ν and Krull valuations, also denoted $\vec{v}$, satisfying $R_{\vec{v}} \subsetneq R_\nu$. These valuations are exactly the exceptional curve valuations centered at points on the exceptional divisor associated to ν.*

Remark 11.8. The space of Krull valuations $\vec{v}$ with $R_{\vec{v}} \subsetneq R_\nu$ is canonically isomorphic to the space of Krull valuations on the residue field k_ν. As $k_\nu \simeq \mathbf{C}(T)$, this space of valuations is isomorphic to $\mathbf{P}^1$ with the Zariski topology, but this identification is not canonical. Moreover, the weak (or strong) topology on $\mathcal{V}$ induces the *discrete* topology on the tree tangent space.

Let us first prove a preliminary result.

Lemma 11.9. *Let $\nu \in \mathcal{V}$ be divisorial, $I = [0, \varepsilon]$ an interval and $I \ni t \mapsto \nu_t \in \mathcal{V}$ a mapping with $\nu_0 = \nu$ such that $\lambda := \frac{d}{dt}\alpha(\nu_t)$ is constant and non-zero on I. Consider $\mathbf{R} \times \mathbf{R}$ with the lexicographic order. Then the function*

$$\nu'(\psi) = \left(\nu(\psi), \frac{1}{\lambda}\frac{d}{dt}\bigg|_{t=0} \nu_t(\psi)\right)$$

defines a centered Krull valuation on R.

Proof. Clearly $\nu'(\phi\psi) = \nu'(\phi) + \nu'(\psi)$ for $\phi, \psi \in R$. Further, $\nu_t(\phi + \psi) \geq \min\{\nu_t(\phi), \nu_t(\psi)\}$ for any t. If strict inequality holds at $t = 0$, then $\nu'(\phi + \psi) \geq \min\{\nu'(\phi), \nu'(\psi)\}$ by the lexicographic ordering. Otherwise $t \mapsto \nu_t(\phi + \psi)$, and $t \mapsto \min\{\nu_t(\phi), \nu_t(\psi)\}$ are affine functions near $t = 0$, coinciding at $t = 0$. Then the slope of the latter cannot exceed the slope of the former, and this implies that $\nu'(\phi + \psi) \geq \min\{\nu'(\phi), \nu'(\psi)\}$. This completes the proof. ☐

Proof of Theorem 11.7. Consider a a tree tangent vector $\vec{v}$ at ν. This is represented by a segment that can be parameterized by skewness. Lemma 11.9 gives us a Krull valuation, also denoted by $\vec{v}$. It does not depend on the choice of the segment, as two segments defining the same tree tangent vector intersect in a one-sided neighborhood of ν, hence define the same Krull valuation.

It is clear by construction that $R_{\vec{v}} \subset R_\nu$. Let us show that $R_{\vec{v}} \subsetneq R_\nu$, that is $\vec{v}$ and ν are not equivalent. For simplicity we assume that $\vec{v}$ is not represented $\nu_{\mathfrak{m}}$, but by some $\nu_\phi > \nu$, where $\phi \in \mathfrak{m}$ is irreducible. Then

$$\vec{v}(\psi) = \left(\nu(\psi), \frac{d}{dt}\bigg|_{t=0} \nu_{\phi, t+\alpha(\nu)}(\psi)\right), \tag{11.1}$$

for any $\psi \in \mathfrak{m}$. Pick $\psi \in \mathfrak{m}$ irreducible with $\nu_\phi \wedge \nu_\psi = \nu$ and set $\chi = \psi^m/\phi^n$, where $m = m(\phi)$, $n = m(\psi)$. Then (11.1) shows that that $\vec{v}(\chi) = (0, -n) < 0$, whereas $\nu(\chi) = 0$. Hence $R_{\vec{v}} \subsetneq R_\nu$. By Lemma 1.4 this implies that $\vec{v}$ is an exceptional curve valuation.

The same argument shows that the assignment $\vec{v} \to R_{\vec{v}}$ from tree tangent vectors to exceptional curve valuations centered at E, is injective.

Conversely, by Lemma 1.4 any valuation $\vec{v}$ with $R_{\vec{v}} \subsetneq R_\nu$ is equivalent to an exceptional curve valuation of the form $\vec{v} = (\nu, \mu)$ for some function $\mu : R \to \mathbf{R}$. Let us show that $\vec{v}$ corresponds to a tangent vector. Write $\nu = \mathrm{val}\,[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$. We have $\nu(U_{j+1}) > \nu(U_j^{n_j})$ for $0 \leq j < k$, hence $\vec{v}(U_{j+1}) > \vec{v}(U_j^{n_j})$ also. Thus $\vec{v}$ has the same sequence of key polynomials as ν up to order k.

First suppose that the length $(\vec{v}) > k$, i.e. $\vec{v}(U_{k+1}) > n_k \cdot \vec{v}(U_k)$ for some $U_{k+1} = U_k^{n_k} - \theta_k \cdot \prod U_j^{m_{kj}}$. As $\nu(U_{k+1}) = n_k \cdot \nu(U_k)$, it follows that $\vec{v}(U_{k+1}) - n_k \cdot \vec{v}(U_k) = (0, \gamma)$ for some $\gamma > 0$. In particular, $\vec{v}(U_{k+1}) \notin \sum_0^k \mathbf{Z} \cdot \vec{v}(U_j) \subset \mathbf{Q} \cdot \vec{v}(U_k)$. Hence $\vec{v} = \mathrm{val}\,[(U_j)_0^{k+1}; (\tilde{\beta}_j, \gamma_j)_0^{k+1}]$, with $\tilde{\beta}_{k+1} = n_k\tilde{\beta}_k$, $\gamma_{j+1}\tilde{\beta}_j = \tilde{\beta}_{j+1}\gamma_j$ for $j \leq k$, and $\gamma_{k+1} > n_k\gamma_k$. We may assume that $\nu(x) = (\tilde{\beta}_0, 0)$. Hence $\vec{v}(U_j) = (\tilde{\beta}_j, 0)$ for

$j \leq k$, and $\vec{v}(U_{k+1}) = (n_k\tilde{\beta}_k, \gamma)$ for some $\gamma > 0$. It is then easy to verify that $\vec{v}$ corresponds to the tangent vector represented by $\phi = U_{k+1}$ in the sense of (11.1).

If instead length $(\vec{v}) = k$, the same type of argument shows that $\vec{v}$ corresponds to the tangent vector represented by $\nu_{\mathfrak{m}}$. □

12. The Hausdorff-Zariski topology

We now present a natural refinement of the Zariski topology, the Hausdorff-Zariski topology on $\mathcal{V}_K$. This turns out to be the weak tree topology for a natural **Z**-tree structure on $\mathcal{V}_K$. We compare the two induced tree structures on $\mathcal{V}$.

12.1. **Definition.** Let us embed $\mathcal{V}_K$ inside the set $\mathcal{F}$ of functions from K to $\{0, +, -\}$ by setting $\nu(\phi) = +, -, 0$ iff $\nu(\phi) > 0$, $\nu(\phi) = 0$, $\nu(\phi) < 0$, respectively. The Zariski topology coincides with the product topology associated to the topology on $\{0, +, -\}$ whose open sets are given by $\emptyset$, $\{0, +\}$, $\{0, +, -\}$. We define the *Hausdorff-Zariski topology* on $\mathcal{V}_K$ to be the topology induced by the product topology associated to the discrete one on $\{0, +, -\}$. This is Hausdorff by construction and we have

Lemma 12.1. *A sequence $\nu_n \in \mathcal{V}_K$ converges towards ν in the Hausdorff-Zariski topology iff for all $\phi \in K$ with $\nu(\phi) > 0$, $\nu(\phi) = 0$ and $\nu(\phi) < 0$ one has $\nu_n(\phi) > 0$, $\nu_n(\phi)$ and $\nu_n(\phi) < 0$, respectively, for sufficiently large n.*

From Tychonov's theorem, and the fact that $\mathcal{V}_K$ is closed in $\mathcal{F}$ (see [ZS]) one deduces the following fundamental result.

Theorem 12.2 ([ZS]). *The space $\mathcal{V}_K$ endowed with the Hausdorff-Zariski topology is compact. Thus $\mathcal{V}_K$ is quasi-compact in the Zariski topology.*

12.2. **The Z-tree structure on $\mathcal{V}_K$.** We now introduce a tree structure on $\mathcal{V}_K$ which plays the same role for the HZ topology as the **R**-tree structure plays for the weak topology on $\mathcal{V}$. The new tree will be modeled on the totally ordered set $\mathbf{N} \cup \{\infty\}$ (see Section 6.1), although we will refer to it as a **Z**-tree.

Definition 12.3. Define a partial order $\trianglelefteq$ on $\mathcal{V}_K$ by

$$\nu_1 \trianglelefteq \nu_2 \text{ iff the sequence of blow-ups } \Pi[\nu_2] \text{ contains } \Pi[\nu_1].$$

Proposition 12.4. *The space $(\mathcal{V}_K, \trianglelefteq)$ is a $\mathbf{Z}$-tree rooted at $\nu_{\mathfrak{m}}$. Any divisorial valuation is a branch point, with tangent space in bijection with $\mathbf{P}^1$. The ends of the tree are exactly the non-divisorial valuations.*

Proof. That $(\mathcal{V}_K, \trianglelefteq)$ is a **Z**-tree (or rather $\mathbf{N} \cup \{\infty\}$-tree) is a straightforward consequence of the definition, as is the fact that the set of ends coincides with the set of non-divisorial valuations.

A tangent vector at a divisorial valuation ν is given by a point on the exceptional component E defining ν. Hence the tangent space at ν is in bijection with $E \simeq \mathbf{P}^1$. This completes the proof. □

Proposition 12.5. *The Hausdorff-Zariski topology coincides with the weak tree topology induced by $\trianglelefteq$.*

Proof. Pick a tangent vector $\vec{v}$ at a divisorial valuation ν_0, and consider the weak open set $V_{\vec{v}}$. If ν_0 corresponds to the exceptional curve E, and $\vec{v}$ to the point $p \in E$, $V_{\vec{v}}$ coincides with the set of valuations that are centered at p. Choose $\phi, \phi' \in K$ defining two smooth transversal curves at p disjoint from E. Then $V_{\vec{v}} = \{\nu \mid \nu(\phi), \nu(\phi') > 0\}$ is a weak tree open set. Hence the identity map from $\mathcal{V}_K$ endowed with the HZ-topology onto $\mathcal{V}_K$ endowed with the weak tree topology associated to $\triangleleft$ is continuous. These two topological spaces are compact, hence homeomorphic. □

Appendix C contains a comparison of **Z**-tree and **R**-tree structure on $\mathcal{V}$ associated to $\trianglelefteq$ and $\leq$, respectively (see the proof of Lemma C.4). For example, consider the segment $I = [\nu_{\mathfrak{m}}, \nu_y]$ in $\mathcal{V}$, consisting of monomial valuations with $1 = \nu(x) \leq \nu(y)$. The restriction of $\leq$ to I coincides with the natural order on $[1, \infty]$, whereas $\trianglelefteq$ gives the lexicographic order on the continued fractions expansions of elements in $[1, \infty]$.

13. Comparison of topologies

We now have three topologies on valuation space $\mathcal{V}$: the weak topology, the strong topology and the restriction of the Hausdorff-Zariski topology to $\mathcal{V} \subset \mathcal{V}_K$. The latter will be referred to as the HZ topology.

Theorem 13.1. *The strong topology and the HZ topology are both stronger than the weak topology. Moreover:*

(i) *if ν has infinite skewness and $\nu_n \to \nu$ weakly, then $\nu_n \to \nu$ strongly;*
(ii) *if ν is non-divisorial and $\nu_n \to \nu$ weakly, then $\nu_n \to \nu$ in the HZ topology.*

Note that no other implications hold as the following examples indicate.

Example 13.2. Let $\nu_n := \nu_{y,1+n^{-1}}$. Then $\nu_n \to \nu_{\mathfrak{m}}$ strongly (hence weakly). But $\nu_n(y/x) = 1/n > 0$ and $\nu_{\mathfrak{m}}(y/x) = 0$, so $\nu_n \not\to \nu$ in the HZ topology. In fact, it converges to the exceptional curve valuation $\mathrm{val}\,[(x,y);((1,0),(1,1))]$.

In this example, the HZ limit valuation is in the closure of $\nu_{\mathfrak{m}}$ in $\mathcal{V}_K$. This is a general fact for limits of sequences in the weak and HZ topology.

Example 13.3. Pick p_n, q_n relatively prime with $p_n/q_n \to \sqrt{2}$. Set $\phi_n = y^{q_n} - x^{p_n}$, $\nu_n = \nu_{\phi_n}$ and $\nu = \nu_{y,\sqrt{2}}$. Then $\nu_n \to \nu$ weakly but $d_{\mathcal{V}}(\nu_n, \nu) \to 1/\sqrt{2}$ so $\nu_n \not\to \nu$ strongly.

Proof of Theorem 13.1. We know from Proposition 10.2 that the strong topology is stronger than the weak topology. The natural injection $\jmath : (\mathcal{V}, \mathrm{HZ}) \to (\mathcal{V}_K, \mathrm{Z})$ is continuous as the HZ topology is stronger than the Zariski topology. By Theorem 11.6 there is a continuous mapping $q : (\mathcal{V}_K, \mathrm{Z}) \to (\mathcal{V}, \mathrm{weak})$ which is the identity on $\mathcal{V} \subset \mathcal{V}_K$. Since $q \circ \jmath = \mathrm{id}$, we see that $\mathrm{id} : (\mathcal{V}, \mathrm{HZ}) \to (\mathcal{V}, \mathrm{weak})$ is continuous. Hence the HZ topology is stronger than the weak topology.

For (i), suppose $\nu_n \to \nu$ weakly. As $\alpha(\nu) = \infty$ we may find $\phi_k \in \mathfrak{m}$ such that $\nu(\phi_k) \geq (k+1)m(\phi_k)$. For $n \gg 1$, $\nu_n(\phi_k) \geq k\, m(\phi_k)$. Thus $\nu, \nu_n \geq \nu_{\phi_k,k}$, so

$$d_{\mathcal{V}}(\nu_n, \nu) \leq d_{\mathcal{V}}(\nu_n, \nu_{\phi_k,k}) + d_{\mathcal{V}}(\nu_{\phi_k,k}, \nu) \leq k^{-1} + k^{-1}.$$

We let $k \to \infty$ and conclude that ν_n converges strongly towards ν.

For (ii), suppose $\nu_n \to \nu$ weakly. Consider $\phi \in K$. If $\nu(\phi) > 0$ (< 0), then $\nu_n(\phi) > 0$ (< 0) for $n \gg 1$. If $\nu(\phi) = 0$, then as the residue field k_ν is isomorphic to $\mathbf{C}$, we may write $\phi = \lambda + \psi$, where $\lambda \in \mathbf{C}^*$ and $\nu(\psi) > 0$. For $n \gg 1$ $\nu_n(\psi) > 0$ so that $\nu_n(\phi) = 0$. Thus $\nu_n \to \nu$ in HZ. $\square$

Part 5. Valuations and curves

In this part we describe how toroidal valuations can be identified with balls of irreducible curves in a particular (ultra-)metric. This allows us to extend concepts such as multiplicities and intersection multiplicities to valuations.

14. Valuations through intersections

The starting point is the observation that a curve valuation acts by intersection (see (E4) in Section 1.3). More precisely, given $\phi, \psi \in \mathfrak{m}$ let $\phi \cdot \psi := \dim_{\mathbf{C}}(R/(\phi, \psi))$ denote the *intersection multiplicity* between ϕ and ψ. Then $m(\phi) = \phi \cdot \psi$ for a "generic" smooth ψ and if ϕ is irreducible, then for any $\psi \in \mathfrak{m}$:

$$\nu_\phi(\psi) = \frac{\phi \cdot \psi}{m(\phi)}. \tag{14.1}$$

Spivakovsky [Sp] showed that a divisorial valuation ν acts (up to normalization) by intersection with a ν-generic curve. If ν is associated to the exceptional divisor E of a sequence of blow-ups π, a ν-generic curve is by definition an irreducible curve whose strict transform under π is smooth, transversal to E and intersects E at a smooth point. By Corollary C.6 below, any ν-generic ϕ satisfies $\nu_\phi \geq \nu$.

Here we extend Spivakovsky's result to a general toroidal valuation.

Proposition 14.1. *If $\nu \in \mathcal{V}$ is toroidal and $\psi \in \mathfrak{m}$ then*

$$\begin{aligned} \nu(\psi) &= \min\{\nu_\phi(\psi) \mid \phi \in \mathfrak{m} \text{ irreducible}, \ \nu_\phi \geq \nu\} \\ &= \min\left\{ \frac{\phi \cdot \psi}{m(\phi)} \;\middle|\; \phi \in \mathfrak{m} \text{ irreducible}, \ \nu_\phi \geq \nu \right\}. \end{aligned}$$

Proof. By (14.1) we only have to show the first equality. Pick $\psi \in \mathfrak{m}$. The inequality $\nu(\psi) \leq \nu_\phi(\psi)$ for $\nu_\phi \geq \nu$ is trivial. For the other inequality write $\psi = \psi_1 \cdots \psi_n$, with $\psi_i \in \mathfrak{m}$ irreducible. First assume that ν is divisorial. Pick a tangent vector $\vec{v}$ at ν which is not represented by any ψ_i nor $\nu_{\mathfrak{m}}$, and choose $\phi \in \mathfrak{m}$ irreducible representing $\vec{v}$. Then $\nu_\phi \geq \nu$ and $\nu_\phi \wedge \nu_{\psi_i} = \nu$ for all i. so by Proposition 8.4 we get $\nu(\psi_i) = \nu_\phi(\psi_i)$ so that $\nu(\psi) = \nu_\phi(\psi)$.

Thus Proposition 14.1 holds for ν divisorial. But if $\nu \in \mathcal{V}$ is irrational, then we may pick ν_n divisorial with $\nu_n > \nu$ and $\nu_n \to \nu$ (say weakly). Fix $\psi \in \mathfrak{m}$. For each n there exists $\phi_n \in \mathfrak{m}$ irreducible with $\nu_{\phi_n} > \nu_n > \nu$ such that $\nu_{\phi_n}(\psi) = \nu_n(\psi)$. Since $\nu_n(\psi) \to \nu(\psi)$ we obtain the desired equality. ☐

15. Balls of curves

Proposition 14.1 shows that a toroidal valuation ν is determined by the irreducible curves ϕ satisfying $\nu_\phi \geq \nu$. We proceed to show that this gives an isometry between the subtree of toroidal valuations, and the set of balls of irreducible curves in a particular (ultra)-metric. Namely, let $\mathcal{C}$ be the set of local irreducible curves. The map $\phi \mapsto \nu_\phi$ naturally embeds $\mathcal{C}$ in $\mathcal{V}$. Define $d_{\mathcal{C}}(\phi, \psi) = \frac{1}{2} d_{\mathcal{V}}(\nu_\phi, \nu_\psi)$.

Lemma 15.1. [G, Corollary 1.2.3] *For $\phi_1, \phi_2 \in \mathcal{C}$, we have*

$$d_{\mathcal{C}}(\phi_1, \phi_2) = \frac{1}{2} d_{\mathcal{V}}(\nu_\phi, \nu_\psi) = \frac{1}{\alpha(\nu_{\phi_1} \wedge \nu_{\phi_2})} = \frac{m(\phi_1) m(\phi_2)}{\phi_1 \cdot \phi_2}. \tag{15.1}$$

Further, $d_{\mathcal{C}}$ is an ultrametric on $\mathcal{C}$; the diameter of $(\mathcal{C}, d_{\mathcal{C}})$ is 1.

Proof. Let us first show (15.1). The first equality holds by definition and the second follows from (8.4) as ν_{ϕ_1} and ν_{ϕ_2} have infinite skewness. For the third equality we will make repeated use of Proposition 8.4. Assume $\nu_{\phi_1} \neq \nu_{\phi_2}$ and set $\nu = \nu_{\phi_1} \wedge \nu_{\phi_2}$. This is divisorial and $\alpha(\nu) = \nu(\phi_1)/m(\phi_1)$ as $\nu_{\phi_1} > \nu$. But $\nu(\phi_1) = \nu_{\phi_2}(\phi_1) = \phi_1 \cdot \phi_2/m(\phi_2)$. Thus the last equality of (15.1) holds.

The fact that $d_{\mathcal{C}}$ is an ultrametric for which $\mathcal{C}$ has diameter 1 follows easily from the formula $d_{\mathcal{C}}(\phi_1, \phi_2) = \alpha^{-1}(\nu_{\phi_1} \wedge \nu_{\phi_2})$ and the tree structure of $\mathcal{V}$. The details are left to the reader. □

In Section 6.4 we constructed a rooted metric tree associated to an ultrametric space of diameter 1. Let $\mathcal{T}_{\mathcal{C}}$ be the metric tree associated to $(\mathcal{C}, d_{\mathcal{C}})$. A point in $\mathcal{T}_{\mathcal{C}}$ is a closed ball in $\mathcal{C}$ of positive radius. The order on $\mathcal{T}_{\mathcal{C}}$ is given by reverse inclusion. The distance between two comparable balls $B_1 \subset B_2$ of radius $r_1 \leq r_2$ is $d(B_1, B_2) = r_2 - r_1$. The distance between two arbitrary balls is $d(B_1, B_2) = d(B_1, B_1 \cap B_2) + d(B_2, B_1 \cap B_2)$.

Theorem 15.2. *Let $\mathcal{C}$ be the set of irreducible curves endowed with the ultrametric $d_{\mathcal{C}}$ given by* (15.1)*, and let $\mathcal{T}_{\mathcal{C}}$ be the associated metric tree (see* (6.4)*).*

If $\nu \in \mathcal{V}$ is a toroidal valuation, we set $B(\nu) = \{\phi \in \mathcal{C} \mid \nu_\phi \geq \nu\}$. This defines a ball in $\mathcal{T}_{\mathcal{C}}$. Conversely, if $B \in \mathcal{T}_{\mathcal{C}}$ is a ball we set $\nu_B = \wedge_{\phi \in B} \nu_\phi$.

The map $B \mapsto \nu_B$ is an isometry from $\mathcal{T}_{\mathcal{C}}$ to the subtree $\mathcal{V}_{\mathrm{tor}}$ of $\mathcal{V}$ consisting of toroidal valuations. The inverse is given by $\nu \mapsto B(\nu)$.

Proof. This is essentially a consequence of Proposition 8.6. The main observation is that if B is centered at ϕ and of radius $r \in (0, 1]$, then $\nu_B = \nu_{\phi, r^{-1}}$. Indeed, if $\psi \in \mathcal{C}$, then $\nu_{\phi, r^{-1}}$ and $\nu_\phi \wedge \nu_\psi$ both belong to the segment $[\nu_{\mathfrak{m}}, \nu_\phi]$. Hence

$$d_{\mathcal{C}}(\phi, \psi) \leq r \Leftrightarrow \alpha(\nu_\phi \wedge \nu_\psi) \geq r^{-1} \Leftrightarrow \nu_\phi \wedge \nu_\psi \geq \nu_{\phi, r^{-1}} \Leftrightarrow \nu_\psi \geq \nu_{\phi, r^{-1}}.$$

Since every toroidal valuation is of the form $\nu_{\phi, t}$, this shows that the mappings $B \mapsto \nu_B$ and $\nu \mapsto B(\nu)$ are well defined and each others inverse. The fact that they are in fact tree isomorphisms is easy to verify and left to the reader. □

Remark 15.3. Similarly one may show that open balls in $\mathcal{C}$ correspond to tree tangent vectors at toroidal valuations in $\mathcal{V}$.

16. Multiplicities

The fact that toroidal valuations can be identify with balls of curves allows us to extend certain notions from curves to arbitrary valuations.

We first define the *multiplicity* $m(\nu)$ of a toroidal $\nu \in \mathcal{V}$ to be the integer

$$m(\nu) = \min\{m(\phi) \mid \phi \in \mathfrak{m} \text{ irreducible}, \ \nu_\phi \geq \nu\}.$$

This multiplicity can be extended to nontoroidal valuations by observing that $\nu \mapsto m(\nu)$ is increasing. We also define $m(\vec{v})$ for a tangent vector at a divisorial valuation as follows: if $\vec{v}$ is represented by $\nu_{\mathfrak{m}}$ then $m(\vec{v}) = m(\nu)$. Otherwise $m(\vec{v})$ is the minimum of $m(\phi)$ over all ν_ϕ representing $\vec{v}$.

In terms of STKP's, the multiplicity of $\nu = \text{val}\,[(U_i); (\tilde{\beta}_i)]$ equals the maximum of the degree in y of the polynomials U_i as long as $\nu(y) \geq \nu(x)$. This implies that $m(\nu) = \infty$ if and only if ν is infinitely singular. Moreover

Proposition 16.1. *If $\nu \in \mathcal{V}$ is infinitely singular, then $\nu(K) \subset \mathbf{Q}$ is infinitely generated over $\mathbf{Z}$. Otherwise, $m(\nu) < \infty$ and*

$$\nu(K) \cap \mathbf{Q} = \frac{1}{m(\nu)} \cdot \mathbf{Z}.$$

Here $\nu(K)$ denotes the value group of ν.

We now define a natural intersection product on $\mathcal{V}$ extending the one in $\mathcal{C}$.

Proposition-Definition 16.2. *We define the* intersection multiplicity *between two valuations $\nu_1, \nu_2 \in \mathcal{V}$ to be the number*

$$\nu_1 \cdot \nu_2 := \alpha(\nu_1 \wedge \nu_2).$$

The assignment $\mathcal{V} \times \mathcal{V} \ni (\nu_1, \nu_2) \mapsto \nu_1 \cdot \nu_2 \in [1, \infty]$ is strongly continuous.

When neither ν_1 nor ν_2 are infinitely singular, then

$$\nu_1 \cdot \nu_2 = \inf \left\{ \frac{\phi_1 \cdot \phi_2}{m(\phi_1) m(\phi_2)} \;\middle|\; \phi_i \in \mathfrak{m} \textit{ irreducible}, \ \nu_{\phi_i} \geq \nu_i \right\}. \tag{16.1}$$

In particular, if $\phi_1, \phi_2 \in \mathfrak{m}$ are irreducible, then $\nu_{\phi_1} \cdot \nu_{\phi_2} = \frac{\phi_1 \cdot \phi_2}{m(\phi_1) m(\phi_2)}$.

Proof. The only thing to show is (16.1). If ϕ_i are irreducible curves with $\nu_{\phi_i} \geq \nu_i$, then (14.1) and Proposition 8.4 give

$$\frac{\phi_1 \cdot \phi_2}{m(\phi_1) m(\phi_2)} = \frac{\nu_{\phi_1}(\phi_2)}{m(\phi_2)} = \alpha(\nu_{\phi_1} \wedge \nu_{\phi_2}) \geq \alpha(\nu_1 \wedge \nu_2) \tag{16.2}$$

If ν_1 and ν_2 are not comparable, then equality holds above in (16.2), proving (16.1). Otherwise we assume $\nu_{\phi_1} \geq \nu_1 \geq \nu_2$ and choose a sequence of curves ϕ_2^k such that $\nu_{\phi_2^k} \geq \nu_2$ and $\nu_{\phi_1} \wedge \nu_{\phi_2^k}$ converges strongly to ν_2. Then (16.2) implies

$$\frac{\phi_1 \cdot \phi_2^k}{m(\phi_1) m(\phi_2^k)} \to \alpha(\nu_1 \wedge \nu_2),$$

which completes the proof. □

17. Classification

Let us conclude this part by the following summarizing table.

Type		rk	rat.rk	tr.deg	skew	mult	Tree term
Toroidal	Divisorial	1	1	1	$< \infty$	$< \infty$	Branch point
	Irrational		2	0			Regular point
Curve	Nonexceptional	2	2	0	∞	$< \infty$	End
	Exceptional				$< \infty$		Tangent vector
Infinitely singular		1	1	0	$\leq \infty$	∞	End

TABLE 1. Elements of valuation space.

Appendix. Alternative approaches to the valuative tree

In this appendix we outline two alternative routes to the valuative tree $\mathcal{V}$. The first one is based on Puiseux series. Any local irreducible curve, save for $x = 0$, is represented by a (non-unique) Puiseux series in x; similarly we can represent a toroidal valuation by a ball of Puiseux series. This approach is closely related to Berkovich's theory of analytic spaces and we show how $\mathcal{V}$ naturally embeds (as a nonmetric tree) as the closure of a disk in the Berkovich projective line $\mathbb{P}^1(k)$ over the local field $k = \mathbf{C}((x))$.

The second approach is to view $\mathcal{V}$ as the universal dual graph of sequences of point blow-ups above the origin. This graph is a tree isomorphic to $\mathcal{V}_{\mathrm{tor}}$.

Both approaches lead to the same natural metric tree structure on $\mathcal{V}_{\mathrm{tor}}$. In the case of Puiseux series it derives from radii of balls or, equivalently, from an identification of a subset of the Berkovich projective line with the Bruhat-Tits building of PGL_2. In the case of the universal dual graph, the metric stems from a natural *Farey parameterization*. We call this tree metric the *thin metric*. It is obtained from an invariant of valuation the *thinness*. The thinness is obtained from skewness after multiplication with an integer-valued weight.

As references for this appendix we point out [T1] for Puiseux expansions, and [B] and [BT] for Berkovich spaces and Bruhat-Tits buildings, respectively; see also [RL] for a concrete exposition in a similar, p-adic, context.

A. Thinness

We define the *thinness* of a valuation $\nu \in \mathcal{V}$ by

$$A(\nu) = 2 + \int_{\nu_{\mathfrak{m}}}^{\nu} m(\mu)\, d\alpha(\mu).$$

Concretely for ν toroidal there exists a unique, strictly increasing sequence $(\nu_i)_{i=0}^{g}$ of divisorial valuations with $\nu_0 = \nu_{\mathfrak{m}}$, $\nu_g < \nu_{g+1} := \nu$ such that $m(\mu)$ is constant, $m(\mu) \equiv m_i$ for $\mu \in]\nu_i, \nu_{i+1}]$. Then $A(\nu) = 2 + \sum_{i=0}^{g} m_i(\alpha_{i+1} - \alpha_i)$. Here $\alpha_i = \alpha(\nu_i)$. This formula extends naturally to nontoroidal valuations.

Notice that if $\nu' < \nu < \nu_\phi$ and $m(\nu_\phi) = m(\nu')$, then $A(\nu) - A(\nu') = \nu(\phi) - \nu'(\phi)$.

We have $A(\nu) \geq 1 + \alpha(\nu)$ with equality iff $m(\nu) = 1$. Clearly $A(\nu) < \infty$ for ν toroidal and $A(\nu) = \infty$ for curve valuations ν. When ν is infinitely singular $A(\nu)$ can be any number in $(2, +\infty]$.

Remark A.1. The name "thinness" was chosen with the following picture in mind. If $\nu = \nu_{\phi,t}$ is toroidal, and $r > 0$ is small, then the region

$$\mathcal{A}_r = \left\{ (x, y) \in \mathbf{C}^2 \mid |(x, y)| \leq r, |\phi(x, y)| \leq |(x, y)|^{tm(\phi)} \right\} \subset \mathbf{C}^2$$

is a small neighborhood of the curve $\phi = 0$. A large value of t (and hence $A(\nu)$) corresponds to a "thin" neighborhood. In fact the diameter and volume of $\mathcal{A}_r$ are roughly r and $r^{2A(\nu)}$, respectively. This point of view is important in [FJ2].

Example A.2. Define $\tilde{\beta}_k = (4^{k+1} - 1)/(3 \cdot 2^k)$, and $\theta_k = 1$ for $k \geq 1$. Then $n_k = 2$ and we define U_k by induction using (P2) of Section 2. Set $\nu = \mathrm{val}\,[(U_k); \tilde{\beta}_k]$.

This is an infinitely singular valuation, and we have

$$A(\nu) = 2 + \sum (\tilde{\beta}_j - 2\tilde{\beta}_{j-1}) = 2 + \sum_1^{\infty} 2^{-j} < \infty.$$

There is a unique tree metric D on $\mathcal{V}_{\mathrm{tor}}$, the *thin metric* satisfying $D(\nu_1, \nu_2) = A(\nu_1) - A(\nu_2)$ when $\nu_1 \geq \nu_2$. The completion of $\mathcal{V}_{\mathrm{tor}}$ with respect to D is the union of $\mathcal{V}_{\mathrm{tor}}$ and infinitely singular valuations with finite thinness.

B. Valuations through Puiseux series

We have seen how to parameterize valuations with irreducible curves and real numbers. Irreducible curves have Puiseux expansions and this leads to an alternative approach for classifying valuations.

B.1. **Puiseux series and valuations.** Let $k = \mathbf{C}((x))$ be the fraction field of $\mathbf{C}[[x]]$ and $\hat{k}$ its algebraic closure, the elements of which are of the form

$$\hat{\phi} = \sum_{j \in \mathbf{N}^*} a_j x^{\hat{\beta}_j}, \quad \text{with} \quad a_j \in \mathbf{C},\ \hat{\beta}_{j+1} > \hat{\beta}_j \in \mathbf{Q} \qquad \text{(B.1)}$$

and $m\hat{\beta}_j \in \mathbf{Z}$ for all j for some integer m. We endow k and $\hat{k}$ with the standard valuation $\nu_\star|_{\mathbf{C}^*} = 0$, and $\nu_\star(x) = 1$, and we let $\bar{k}$ be the completion of $\hat{k}$ with respect to $\nu_\star$. Its elements are of the form (B.1) with $\hat{\beta}_j \to \infty$.

Let $\widehat{\mathcal{V}}$ be the set of valuations $\hat{\nu} : \bar{k}[y] \to \overline{\mathbf{R}}_+$ extending $\nu_\star$ on $\bar{k}$ and satisfying $\hat{\nu}(y) > 0$. To $\widehat{\mathcal{V}}$ we also add the valuation $\hat{\nu}_\star$ defined by $\hat{\nu}_\star(y - \hat{\phi}) = 0$ for all $\hat{\phi} \in \bar{k}$. There is a natural partial ordering $\leq$ on $\widehat{\mathcal{V}}$ with minimal element $\hat{\nu}_\star$. As $\overline{k}$ is algebraically closed, any $\hat{\nu} \in \widehat{\mathcal{V}}$ is determined by its values on $y - \hat{\phi}$ over $\hat{\phi} \in \bar{k}$.

Proposition B.1. *If $\hat{\nu} \in \widehat{\mathcal{V}}$, $\hat{\nu} \neq \hat{\nu}_\star$, then there exist $\hat{\beta} > 0$ and a series $\hat{\phi} = \sum a_i x^{\hat{\beta}_i}$ with $a_i \neq 0$, $\hat{\beta}_{i+1} > \hat{\beta}_i > 0$ such that*

$$\hat{\nu}(y - \hat{\psi}) = \min\{\hat{\beta}, \nu_\star(\hat{\psi} - \hat{\phi})\} \qquad \text{(B.2)}$$

for all $\hat{\psi} \in \bar{k}$. The couple $(\hat{\phi}, \hat{\beta})$ is unique as long as $\hat{\beta} > \hat{\beta}_i$ for all i, and $\hat{\beta} = \lim_i \hat{\beta}_i$ if this limit is finite.

Sketch of proof. As noted above, $\hat{\nu}$ is determined by its values on $y - \hat{\phi}$, $\hat{\phi} \in \bar{k}$. Write $\hat{\nu}(y) = \hat{\beta}_1 > 0$. Then $\hat{\nu}(y - \hat{\phi}) \geq \min\{\hat{\beta}_1, \nu_\star(\hat{\phi})\}$, and either equality holds for all $\hat{\phi} \in \bar{k}$, or $\hat{\nu}(y - \hat{\phi}_1) = \hat{\beta}_2 > \hat{\beta}_1 \in \mathbf{Q}$, $\hat{\phi}_1 = \theta_1 x^{\hat{\beta}_1}$ for a unique $\theta_1 \in \mathbf{C}^*$. Inductively we construct $\hat{\phi}_j = \hat{\phi}_{j-1} + \theta_j x^{\hat{\beta}_j}$ with $\theta_j \in \mathbf{C}^*$ and $\hat{\beta}_{j+1} > \hat{\beta}_j \in \mathbf{Q}$, such that $\hat{\nu}(y - \hat{\phi}) \geq \min\{\hat{\beta}_{j+1}, \nu_\star(\hat{\phi} - \hat{\phi}_j)\}$. If this process stops at some finite step i, then we set $\hat{\phi} = \hat{\phi}_i = \sum_1^i \theta_j x^{\hat{\beta}_j}$, $\hat{\beta} = \hat{\beta}_{i+1}$. Otherwise $(\hat{\beta}_j)_{j \geq 1}$ is strictly increasing with limit $\hat{\beta}$ and $\hat{\phi}_j$ converges to a series $\hat{\phi}$ (not necessarily in $\bar{k}$). Equation (B.2) and uniqueness are easily verified. □

Definition B.2. For $\hat{\phi} = \sum a_i x^{\hat{\beta}_i}$ with $a_i \neq 0$, $\hat{\beta}_{i+1} > \hat{\beta}_i$, and $\hat{\beta} > 0$, let $\mathrm{val}\,[\hat{\phi}; \hat{\beta}]$ be the valuation $\hat{\nu} \in \widehat{\mathcal{V}}$ satisfying (B.2). Set $\mathrm{val}\,[\hat{\phi}; 0] = \hat{\nu}_\star$ for any $\hat{\phi}$.

Definition B.3. Let $\hat\nu = \mathrm{val}\,[\hat\phi;\hat\beta] \in \widehat{\mathcal V}$. If $\hat\phi \in \bar k$ and $\hat\beta = \infty$ then $\hat\nu =: \hat\nu_{\hat\phi}$ is of *point type.* If $\hat\phi \in \hat k$ but $\hat\beta < \infty$ then $\hat\nu$ is of *finite type* and *(ir)rational* if $\hat\beta$ is (ir)rational. If $\hat\phi = \sum a_i x^{\hat\beta_i} \notin \bar k$ and $\hat\beta > \hat\beta_i$ then $\hat\nu$ is of *special type.*

Corollary B.4. *The partial ordering $\le$ gives $\widehat{\mathcal V}$ the structure of a complete tree rooted at $\hat\nu_\star$. Its ends are the valuations of point type or special type. Its branch points are the rational valuations.*

Corollary B.5. *The parameterization* $[0,\infty[\ni \hat\beta \mapsto \mathrm{val}\,[\hat\phi;\hat\beta] \in [\hat\nu_\star, \hat\nu_{\hat\phi}[$ *induces a metric tree structure on the set $\widehat{\mathcal V}_{\mathrm{fin}}$ of valuations of finite type.*

We call this metric on $\widehat{\mathcal V}_{\mathrm{fin}}$ the *Puiseux metric.* As in the case of $\mathcal V$ it induces an ultrametric on $\hat k$ and we may view a valuation in $\widehat{\mathcal V}$ of finite type as a ball of Puiseux series.

B.2. **Two trees.** We can relate the two trees $\mathcal V$ and $\widehat{\mathcal V}$ by noting that any valuation on $k[y]$ extending $\nu_\star$ extends, non-uniquely, to a valuation on $\bar k[y]$. For example, the valuations $\mathrm{val}\,[y \pm x^{3/2}; 5/2]$ on $\bar k[y]$ both restrict to $\nu_{y^2-x^3,2}$ on $k[y]$. The non-uniqueness is encoded by the action on $\bar k$ of the Galois group $\mathrm{Gal}(\hat k/k)$, which can be identified with the group $\mathbf U$ of roots of unity: if $\omega \in \mathbf U_m$ is a primitive m-th root of unity and $\beta > 0$ is rational, then $\omega_*(x^\beta) = \omega^p x^\beta$, where $\beta = p/q$ with p, q minimal such that $m|q$. The natural induced action on valuations satisfies $\omega^*\nu_\star = \nu_\star$.

Consider a series $\hat\phi = \sum_{j\ge1} a_i x^{\hat\beta_j}$ with $a_j \in \mathbf C^*$, $\hat\beta_{j+1} > \hat\beta_j \in \mathbf Q$. Write $\hat\beta_j = p_j/q_j$ with $\gcd(p_j,q_j) = 1$ and define the *multiplicity* $m(\hat\phi)$ of $\hat\phi$ to be the number $\mathrm{lcm}(q_j) \in \mathbf N \cup \{\infty\}$. Clearly $m(\hat\phi) < \infty$ iff $\hat\phi \in \hat k$ and in this case the minimal polynomial of $\hat\phi$ over k is given by $\phi(x,y) = \prod_{j=0}^{m-1}(y - \omega_*^j(\hat\phi))$ where $m = m(\hat\phi)$ and ω is a primitive m-th root of unity. If $\hat\beta_1 \ge 1$, then $m(\hat\phi) = m(\phi)$

As with $\mathcal V$ we can extend the notion of multiplicities from elements in $\bar k$ to valuations in $\widehat{\mathcal V}$ of finite type, using the tree structure. Namely, we let $m(\hat\nu) = \min\{m(\hat\phi) \mid \hat\nu \le \hat\nu_{\hat\phi}\}$ and define $\mathbf U(\hat\nu)$ to be the the quotient group $\mathrm{Gal}(\hat k/k)/G$ where $G = \{\omega \mid \omega^*\hat\nu = \hat\nu\}$. We leave to the reader to check that $\mathbf U(\hat\nu)$ is isomorphic to $\mathbf U_{m(\hat\nu)} \subset \mathbf U$. Note that $\hat\nu \mapsto m(\hat\nu)$ and $\hat\nu \mapsto \mathbf U(\hat\nu)$ are both increasing functions on $\widehat{\mathcal V}_{\mathrm{fin}}$ so we may define $\mathbf U(\hat\nu)$ also for $\hat\nu \in \widehat{\mathcal V}$ not of finite type.

We define a natural mapping $\Phi : \widehat{\mathcal V}_{\mathrm{fin}} \to \mathcal V_{\mathrm{tor}}$ letting $\Phi(\hat\nu)$ be the unique normalized valuation equivalent to the restriction $\hat\nu|_{k[y]}$. When $\hat\nu = \mathrm{val}\,[\hat\phi; s]$ for $\hat\phi \in \hat k$, $s > 0$, then ν is the restriction of $\hat\nu/\min\{1, s, \nu_\star(\hat\phi)\}$ to $k[y]$. In the following theorem $\mathcal V$ is equipped with the partial ordering $\preceq$ rooted at the curve valuation ν_x. On the subtree $\{\nu \succeq \nu_{\mathfrak m}\}$ this ordering coincides with the usual partial ordering $\le$ rooted at $\nu_{\mathfrak m}$. We also let $\hat\nu_{\mathfrak m} = \mathrm{val}\,[0;1]$.

Theorem B.6. *The map $\Phi : \widehat{\mathcal V}_{\mathrm{fin}} \to \mathcal V_{\mathrm{tor}}$ is a surjective tree map. More precisely:*

(i) Φ *restricts to an order preserving bijection of the segment* $]\nu_\star, \hat\nu_{\hat\phi}[$ *onto the segment* $]\nu_x, \nu_\phi[$*;*

(ii) *if* $\hat\nu, \hat\mu \in \widehat{\mathcal{V}}$*, then* $\Phi(\hat\nu) = \Phi(\hat\mu)$ *iff* $\mathbf{U}(\hat\nu) = \mathbf{U}(\hat\mu)$ *and there exists* $\omega \in \mathbf{U}(\hat\nu)$ *such that* $\omega^*\hat\nu = \hat\mu$*;*

(iii) *if* $\hat\nu = \operatorname{val}[\hat\phi; s]$ *and* $s \geq \nu_\star(\hat\phi) \geq 1$*, then* $m(\Phi(\hat\nu)) = m(\nu)$ *and the thinness of* $\Phi(\nu)$ *equals* $s + 1$*.*

In particular, the Puiseux metric on $\widehat{\mathcal{V}}$ *restricts to a tree metric on* $\mathcal{V}$*, and the latter coincides with the thin metric* D *on the subtree* $\{\nu \succeq \nu_{\mathfrak{m}}\} \subset \mathcal{V}$*.*

Note that Φ extends uniquely to a surjective tree map of $\widehat{\mathcal{V}}$ onto $\mathcal{V}$.

Remark B.7. The proof below makes use of the fact, proved in Part 3, that $\mathcal{V}$ is a tree. Alternatively, the analysis in this section could be used to prove the tree structure on $\mathcal{V}$.

Proof. Notice that Φ is continuous in the weak topology by the definition of the latter in terms of pointwise convergence.

Pick $\phi \in \hat{k}$ with $\nu_\star(\hat\phi) > 0$. Set $s_0 = \min\{1, \nu_\star(\hat\phi)\} \in \mathbf{Q}_+$, $\hat\nu_s = \operatorname{val}[\hat\phi; s]$, $\nu_s = \Phi(\hat\nu_s)$. First consider $s \leq s_0$. Then $\hat\nu_s(x) = 1/s$ so if s is irrational, then Corollary 8.7 implies $\nu_s = \nu_{x,1/s}$. By continuity this must hold for all $s \leq s_0$. Now consider $s \geq s_0$ and use the factorization $\phi(x, y) = \prod_j (y - \omega^j_* \hat\phi)$ above. It yields $\nu_s(\phi) = s_0^{-1} \hat\nu_s(\phi) = s_0^{-1} \sum \max\{s, A_j\}$, where $A_0 = 0$ and $A_j > 0$ is rational for $j > 0$. The same argument as above gives $\nu = \nu_{\phi, g(s)}$, where $g(s) := (s_0 m(s))^{-1} \sum \max\{s, A_j\}$. This proves (i).

For (ii), it is clear that for any $\omega \in \operatorname{Gal}(\hat{k}/k)$, the restriction of $\omega^*\hat\nu$ coincides with $\hat\nu$ on $k[y]$. Hence $\Phi(\omega^*\hat\nu) = \hat\nu$. Conversely, suppose $\Phi(\hat\nu) = \Phi(\hat\mu)$. Write $\hat\nu = \operatorname{val}[\hat\phi; s]$ as before. We may assume $s > s_0$ by replacing ϕ with x if necessary. We have $\hat\mu = \operatorname{val}[\omega^*\hat\phi; s']$ for some ω, $s' > 0$, otherwise the map $\nu' \mapsto \nu'(\phi)$ would be locally constant around μ on the segment $[\nu_x, \Phi(\hat\mu)] = [\nu_x, \Phi(\hat\nu)]$ in $(\mathcal{V}, \succeq)$. The previous calculation shows that $t \to \operatorname{val}[\hat\phi; t](\phi)$ is strictly increasing for $t \geq s_0$, hence $\Phi(\hat\nu) = \Phi(\hat\mu)$ implies $s = s'$.

To prove (iii) we need to control the multiplicity and skewness of $\Phi(\hat\nu)$ for $\hat\nu \geq \hat\nu_{\mathfrak{m}}$. We claim that $m(\Phi(\hat\nu)) = m(\hat\nu)$ for $\hat\nu \geq \hat\nu_{\mathfrak{m}}$. Indeed, if $\hat\phi \in \hat{k}$ satisfies $\hat\nu_{\hat\phi} > \hat\nu$ and $m(\hat\phi) = m(\hat\nu)$, then $\nu_\phi \succeq \Phi(\hat\nu)$, where ϕ is the minimal polynomial of $\hat\phi$ over k. Since the order relations $\succeq$ and $\geq$ coincide on $\{\nu \succeq \nu_{\mathfrak{m}}\}$, and $m(\phi) = m(\hat\phi)$, we infer $m(\Phi(\hat\nu)) \leq m(\hat\nu)$. On the other hand, any irreducible $\phi \in \mathfrak{m}$ is the minimal polynomial over k of some $\hat\phi \in \hat{k}$. When $\nu_\phi \geq \nu$, then $\nu_{\omega^*\hat\phi} \geq \hat\nu$ for some ω by what precedes. This gives the reverse inequality.

As for skewness we consider $\hat\nu, \hat\nu' \in \widehat{\mathcal{V}}$ with $\hat\nu_{\mathfrak{m}} \leq \hat\nu' \leq \hat\nu$ and $m(\hat\nu) = m(\hat\nu') =: m$. Pick $\hat\phi \in \hat{k}$ with $\hat\nu_{\hat\phi} > \hat\nu$ and $m(\hat\phi) = m$. Write $\nu = \Phi(\hat\nu)$ and $\nu' = \Phi(\hat\nu')$. Then $\hat\nu = \operatorname{val}[\hat\phi; s]$ and $\hat\nu' = \operatorname{val}[\hat\phi; s']$ for $s' < s$. Let ϕ be the minimal polynomial of $\hat\phi$ over k. We have $\phi = \prod_{\omega \in \mathbf{U}_m} y - \omega_* \hat\phi$. Our assumptions imply that $\nu_\star(\omega_*\hat\phi - \hat\phi) < s'$ for $\omega \neq 1$. Hence

$$A(\nu) - A(\nu') = \nu(\phi) - \nu'(\phi) = \sum_{\omega \in \mathbf{U}_m} \hat\nu(y - \omega_*\hat\phi) - \hat\nu'(y - \omega_*\hat\phi) = s - s'. \quad \text{(B.3)}$$

For $s = 1$ and $\hat{\nu}_{\hat{\phi}} \geq \nu_{\mathfrak{m}}$ we have $\operatorname{val}[\hat{\phi}; s] = \hat{\nu}_{\mathfrak{m}}$ and $A(\Phi(\hat{\nu}_{\mathfrak{m}})) = A(\nu_{\mathfrak{m}}) = 2$. A simple induction based on (B.3) now gives (iii). $\square$

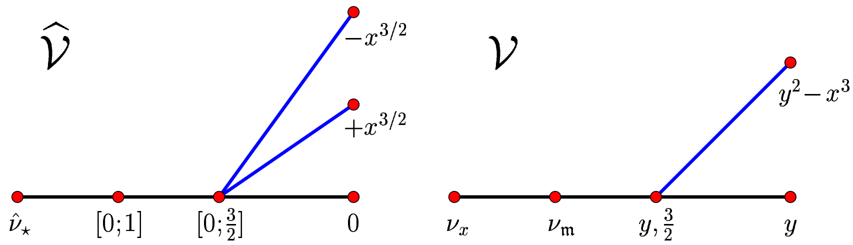

FIGURE 4. The tree structure on the spaces $\widehat{\mathcal{V}}$ and $\mathcal{V}$ and their relation given by Theorem B.6. The notation has been simplified.

B.3. **The Berkovich projective line.** We next indicate how the valuative spaces $\mathcal{V}$ and $\widehat{\mathcal{V}}$ are naturally embedded (as nonmetric trees) in the Berkovich projective lines over the local fields k and $\bar{k}$, respectively.

The Berkovich affine line $\mathbb{A}^1(k)$ is defined ([B] p.19) to be the set of valuations $\nu : k[y] \to (-\infty, +\infty]$ extending $\nu_\star$. The line $\mathbb{A}^1(\bar{k})$ is defined analogously. We will write ν for elements of $\mathbb{A}^1(k)$ and $\hat{\nu}$ for elements of $\mathbb{A}^1(\bar{k})$.

Both lines are endowed with the weak topology. By definition, $\widehat{\mathcal{V}}$ is a subset of $\mathbb{A}^1(\bar{k})$. Berkovich defines ([B] p.18) the open disk in $\mathbb{A}^1(\bar{k})$ with center $\hat{\phi} \in \bar{k}$ and radius $r > 0$ to be $D(\hat{\phi}, r) = \{\hat{\nu} \mid \hat{\nu}(y - \hat{\phi}) > \log r\}$. It follows that $\widehat{\mathcal{V}}$ *is the closure of the open disk in* $\mathbb{A}^1(\bar{k})$ *of radius 1 centered at zero.* Disks in $\mathbb{A}^1(k)$ are defined as images of disks under the restriction map $\mathbb{A}^1(\bar{k}) \to \mathbb{A}^1(k)$. It follows that the disk $D(0, 1)$ in $\mathbb{A}^1(k)$ is the set of valuations $\nu : k[y] \to (-\infty, \infty]$ extending $\nu_\star$ and satisfying $\nu(y) > 0$. But any such valuation becomes a normalized valuation $\nu : \mathbf{C}[[x, y]] \to [0, \infty]$ after dividing with the factor $\min\{1, \nu(y)\}$ and extending. The curve valuation $\nu_x \in \mathcal{V}$ cannot be obtained in this way, but $\mathcal{V}$ *is homeomorphic to the closure of the open disk in* $\mathbb{A}^1(\bar{k})$ *of radius 1 centered at zero.*

The Berkovich projective line $\mathbb{P}^1(k)$ is defined as $\mathbb{A}^1(k) \cup \{\nu_\infty\}$ where $\nu_\infty : k[y] \to [-\infty, \infty]$ equals $\nu_\star$ on k and $-\infty$ elsewhere. The line decomposes as

$$\mathbb{P}^1(k) = \{\nu_\star\} \sqcup \bigsqcup_{\alpha \in \mathbf{P}^1(\mathbf{C})} D_\alpha,$$

where $D_\alpha := D(y - \alpha, 1)$ for $\alpha \in \mathbf{C}$ and $D_\infty := \{\nu \in \mathbb{A}^1(k) \mid \nu(y) < 0\} \cup \{\nu_\infty\}$ and $\nu_\star \in \mathbb{A}^1(k)$ denotes the valuation $\nu_\star(\sum_j a_j y^j) = \min_j \nu_\star(a_j)$.

Any translation $y \mapsto y + \tau$, $\tau \in \mathbf{C}$, induces an isomorphism of $\mathbb{P}^1(k)$ sending D_α homeomorphically to $D_{\alpha+\tau}$. Similarly, the inversion $y \mapsto y^{-1}$ induces a homeomorphism of D_∞ onto D_0. The open disks D_α are hence all homeomorphic, and their closures are $\overline{D_\alpha} = D_\alpha \cup \{\nu_\star\}$.

On any disk $\overline{D_\alpha}$ we can put the tree structure induced by $\mathcal{V}$ rooted at ν_x (see Section 6.1). For instance, on $\overline{D_0}$, the tree structure is given by $\nu \leq \mu$ iff $\nu(\phi) \leq \mu(\phi)$ for all $\phi \in k[y]$. Patched together, these partial orderings endow

$\mathbb{P}^1(k)$ with a natural tree structure rooted at $\nu_\star$ (see [B, Theorem 4.2.1]). By Theorem 9.1, the weak tree topology on $\mathbb{P}^1(k)$ coincides with the weak topology.

This discussion goes through, verbatim, with k replaced by $\bar{k}$.

B.4. **The Bruhat-Tits metric.** The standard projective line $\mathbf{P}^1(\bar{k})$ embeds naturally in the Berkovich line $\mathbb{P}^1(\bar{k})$: $\hat{\phi} \in \bar{k}$ corresponds to the valuation $\hat{\nu}$ with $\hat{\nu}(y - \hat{\phi}) = \infty$ and ∞ to $\hat{\nu}_\infty$. The points in $\mathbf{P}^1(\bar{k})$ are ends of the tree $\mathbb{P}^1(\bar{k})$; hence $\mathbb{H} := \mathbb{P}^1(\bar{k}) \setminus \mathbf{P}^1(\bar{k})$ has a nonmetric tree structure rooted at $\hat{\nu}_\star$. The group $\mathrm{PGL}_2(\bar{k})$ of Möbius transformations acts on $\mathbf{P}^1(\bar{k})$ and this action extends to $\mathbb{P}^1(\bar{k})$ by $(M_*\hat{\nu})(y - \hat{\phi}) = \hat{\nu}(My - \phi)$.

Berkovich noted that $\mathbb{H}$ is isomorphic (as a non-metric tree) to the Bruhat-Tits building of $\mathrm{PGL}_2(\bar{k})$. We will not define this building nor the isomorphism here. Suffice it to say that the building is a metric tree on which $\mathrm{PGL}_2(\bar{k})$ acts by *isometries*. This last condition in fact defines the tree metric on $\mathbb{H}$ up to a constant. To see this, consider the segment in $\mathbb{H}$ parameterized by $\mathrm{val}\,[0;t]$, $0 \le t < \infty$. Fix any rational $t_0 > 0$ and pick $\hat{\phi}_0$ with $\hat{\nu}_\star(\hat{\phi}_0) = t_0$. Then $M \in \mathrm{PGL}_2(\bar{k})$ defined by $My = y\hat{\phi}_0$ gives $M_*\mathrm{val}\,[0;t] = \mathrm{val}\,[0;t+t_0]$. But the only translation invariant metric on $\mathbf{R}$ is the Euclidean metric (up to a constant), so after normalizing we get that $[0,\infty[\, \ni t \mapsto \mathrm{val}\,[0,t] \in \mathbb{H}$ is an isometry. Now if $\hat{\phi} \in \bar{k}$, then $M(y) := y - \hat{\phi}$ induces an isometry of the segment $[\hat{\nu}_\star, \hat{\nu}_{\hat{\phi}}[$ onto $[\hat{\nu}_\star, \hat{\nu}_0[$. Similarly, $M(y) = 1/y$ gives an isometry of $[\hat{\nu}_\star, \infty[$ onto $[\hat{\nu}_\star, 0[$. Thus the metric on $\mathbb{H}$ is uniquely defined.

In particular we see that if $\hat{\phi} \in \bar{k}$ and $\nu_\star(\hat{\phi}) > 0$, then $[0,\infty[\, \ni t \mapsto \mathrm{val}\,[\hat{\phi}, t] \in \mathbb{H}$ is an isometry. Thus *the Puiseux metric on $\widehat{\mathcal{V}}$ is induced by the Bruhat-Tits metric on $\mathbb{H} \supset \widehat{\mathcal{V}}$*. Passing to $\mathcal{V}$ we conclude using Theorem B.6 than *the thin metric on $\mathcal{V}$ is induced by the Bruhat-Tits metric.*

B.5. **Dictionary.** We end this section with a dictionary between valuations in $\mathcal{V}$ (i.e. on $k[y]$), their preimages in $\widehat{\mathcal{V}}$ (i.e. on $\bar{k}[y]$) under the restriction map (see Theorem B.6), and Berkovich's terminology [B, p.18].

<table>
<tr><th colspan="3">Valuations in $\mathcal{V}$</th><th colspan="2">Valuations in $\widehat{\mathcal{V}}$</th><th>Berkovich</th></tr>
<tr><td rowspan="2">Toroidal</td><td colspan="2">Divisorial</td><td rowspan="2">Finite type</td><td>Rational</td><td>Type 2</td></tr>
<tr><td colspan="2">Irrational</td><td>Irrational</td><td>Type 3</td></tr>
<tr><td rowspan="3">Nontoroidal</td><td colspan="2">Curve</td><td rowspan="2">Point type</td><td>$\lvert\mathbf{U}(\nu)\rvert < \infty$</td><td rowspan="2">Type 1</td></tr>
<tr><td rowspan="2">Inf singular</td><td>$A = \infty$</td><td>$\lvert\mathbf{U}(\nu)\rvert = \infty$</td></tr>
<tr><td>$A < \infty$</td><td colspan="2">Special type</td><td>Type 4</td></tr>
</table>

TABLE 2. Terminology

C. The universal dual tree

We finally describe a geometric construction of the valuative tree using dual graphs of sequences of blow-ups. This gives the nonmetric tree structure as an inductive limit of finite simplicial trees. The metric tree structure corresponds to a natural Farey parameterization of dual graphs.

C.1. **Constructions.** To any finite sequence π of blow-ups (not necessarily of infinitely nearby points) is attached a dual graph Γ_π: vertices are in bijection with irreducible components of the exceptional divisor of π, and edges with point of intersection between two components. Thus Γ_π is a finite simplicial tree and can be viewed as a nonmetric tree, still denoted Γ_π, in the sense of Part 3. A natural choice of root is the vertex $\gamma_{\mathfrak{m}}$ corresponding to the (strict transform of) the exceptional divisor from blowing up the origin once.

We can also put a metric tree structure on Γ_π. To this end we first associate to each vertex in Γ_π a vector $(a, b) \in (\mathbf{N}^*)^2$, its *Farey weight*, recursively as follows (compare with [HP]). If π is a single blowup of the origin, then Γ_π is a single point $\gamma_{\mathfrak{m}}$ whose weight is defined to be $(2, 1)$. Otherwise we may write $\pi = \tilde{\pi} \circ \pi'$, where $\tilde{\pi}$ is the blowup of a point p on the exceptional divisor $(\pi')^{-1}(0)$. The Farey weights of the vertices in Γ_π that are strict transforms of vertices in $\Gamma_{\pi'}$ inherit their weights from the latter graph. The only other vertex in Γ_π is the exceptional divisor E_p obtained by blowing up p. The weight of E_p is determined as follows. We say that p is a *free* (*satellite*) point if p is a regular (singular) point on $(\pi')^{-1}(0)$. When p is free, i.e. $p \in E$ for a unique exceptional divisor $E \subset (\pi')^{-1}(0)$, then the weight of E_p is defined to be $(a+1, b)$, where (a, b) is the weight of E. When p is a satellite point, it is the intersection of two components E and E' whose respective weights are (a, b) and (a', b'). The weight of E_p is then $(a + a', b + b')$. See Figure 5.

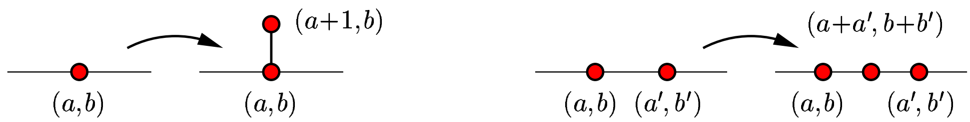

FIGURE 5. Elementary modifications on the dual graph

The *Farey parameter* of a vertex with weight (a, b) is now defined to be the rational number a/b. We leave to the reader check by induction that if E, F are vertices in Γ_π and $E < F$ in the partial order on Γ_π rooted at $\gamma_{\mathfrak{m}}$, then the Farey parameter of E is strictly smaller than that of F. Thus the Farey parameter defines a metric tree structure d_π on Γ_π.

If π is a sequence dominating another sequence π', i.e. $\pi = \tilde{\pi} \circ \pi'$ for some blow-ups $\tilde{\pi}$, then there is a natural map $\imath_{\pi'\pi} : \Gamma_{\pi'} \to \Gamma_\pi$ sending a vertex γ corresponding to a component of $(\pi')^{-1}(0)$ to its strict transform by $\tilde{\pi}$. Thus $\Gamma_{\pi'}$ is naturally a subtree of Γ_π. Moreover $\imath_{\pi'\pi}$ is an isometry. This leads us to define the *universal dual graph* Γ as the inductive limit

$$\Gamma = \varinjlim \Gamma_\pi$$

over all sequences of blow-ups π above the origin. We leave to the reader to verify

Proposition C.1. *The universal dual graph is a rooted metric tree containing no ends. Its branch points are exactly the vertices in some Γ_π and the tree tangent* *space at a branch point is naturally in bijection with* $\mathbf{P}^1$. *The set of branch points further equals the set* $\Gamma_{\mathbf{Q}}$ *of points in* Γ *with rational distance to* $\gamma_{\mathfrak{m}}$ *and is hence strongly dense in* Γ.

A point in $\Gamma_{\mathbf{Q}}$ corresponds to a vertex E in some Γ_π. We may assume that π is the blowup at a finite sequence of infinitely nearby points above the origin and that E is the last exceptional divisor. Hence E defines a divisorial valuation ν_E. This gives a map $\Phi : \Gamma_{\mathbf{Q}} \to \mathcal{V}_{\mathrm{div}}$ that is a bijection by definition. Equip $\mathcal{V}_{\mathrm{tor}}$ with the thin metric D defined in Appendix A.

Theorem C.2. *The bijective map* $\Phi : \Gamma_{\mathbf{Q}} \to \mathcal{V}_{\mathrm{div}}$ *extends uniquely to a tree isomorphism and isometry* $\Phi : \Gamma \to \mathcal{V}_{\mathrm{tor}}$.

C.2. **Proofs.** Since $\Gamma_{\mathbf{Q}}$ and $\mathcal{V}_{\mathrm{div}}$ are strongly dense in Γ and $\mathcal{V}_{\mathrm{tor}}$, respectively, the theorem follows from the following two results.

Lemma C.3. *If* $E \in \Gamma_{\mathbf{Q}}$, *then the thinness of* $\nu_E = \Phi(E)$ *equals the Farey parameter of* E.

Lemma C.4. Φ *is order preserving.*

Before proving these lemmas we need two computational results.

Lemma C.5. *Let* $\nu_E \in \mathcal{V}$ *be a (normalized) divisorial valuation associated to the last exceptional component* E *of a sequence* π *of blowups. For* $\phi \in \mathfrak{m}$, *denote by* $\mathrm{div}_E(\phi)$ *the order of vanishing of* ϕ *along* E. *Then*

$$\pi_* \mathrm{div}_E = b\, \nu_E, \tag{C.1}$$

where (a, b) *is the Farey weight of* E

Proof. The statement holds for $\nu = \nu_{\mathfrak{m}}$. We have to check that (C.1) remains true when ν is replaced by μ obtained by blowing up a point p on E. Suppose first that p is a free point. The Farey weight of μ is then $(a+1, b)$. At the point p, the valuations ν and μ are given by two *normalized* valuations div_E and μ_p, respectively. By assumption $\pi_*\mathrm{div}_E = b\,\nu$. Write $\pi_*\mu_p = b'\,\mu$. We claim that $b = b'$. For a generic ϕ the strict transform of $\phi^{-1}(0)$ by π does not contain the point p, hence $\mu_p(\pi^*\phi) = \mathrm{div}_E(\pi^*\phi)$. Thus

$$b' = b' \min_{\phi \in \mathfrak{m}} \mu_p(\phi) = \min_{\phi \in \mathfrak{m}} \mu_p(\pi^*\phi) = \min_{\phi \in \mathfrak{m}} \mathrm{div}_E(\pi^*\phi) = \min_{\phi \in \mathfrak{m}} b\, \nu_E(\phi) = b.$$

Suppose now that p is a satellite point lying at the intersection of E and another exceptional divisor E'. Let ν' with Farey weight (a', b') be the divisorial valuation associated to E'. The Farey weight of μ is then $(a + a', b + b')$. By assumption $\pi_*\mathrm{div}_E = b\,\nu$, and $\pi_*\mathrm{div}_{E'} = b'\,\nu'$. We proceed as before. For a generic $\phi \in \mathfrak{m}$ $\mu_p(\pi^*\phi) = \mathrm{div}_E(\pi^*\phi) + \mathrm{div}_{E'}(\pi^*\phi)$. We conclude that $\pi_*\mu_p = (b + b')\,\mu$. □

Corollary C.6. *Let* $\nu_E \in \mathcal{V}$ *be divisorial, with associated exceptional divisor* E. *Pick* $\nu \in \mathcal{V}$ *with center at a* smooth *point* $p \in E$. *Then* $\nu_E \le \nu$.

Proof. As in Lemma C.5, denote by ν_p the normalized valuation defined at p by the lift of ν through π. As ν_p is normalized, we have $\nu_p(E) \ge 1$, hence $\nu_p \ge \mathrm{div}_E$. By (C.1) above we infer $\nu \ge \nu_E$. □

Lemma C.7. *Pick a point p infinitely near to the origin. Suppose p is a free point on an exceptional component E with Farey weight (a, b). Then the critical set of the contraction map π at p is given by the divisor $(a-1)[E]$.*

Proof. We show more precisely the following result. At a free point p the critical set of π is given by $(a-1)[E]$, where E is the exceptional component containing p with Farey weight (a, b). At a satellite point $p = E \cap E'$ the critical set is given by $(a-1)[E] + (a'-1)[E']$ with obvious notations.

The proof proceeds by induction on the number of blow-ups. There is nothing to prove when p is the origin in $\mathbf{C}^2$. Suppose the claim has been proved for a point p, let π' be the blow-up at p, and pick a point p' on the exceptional divisor created by π'. The critical divisor of the composition map $\pi \circ \pi'$ is given by

$$[\mathcal{C}_{\pi\circ\pi'}] = \pi'^*[\mathcal{C}_\pi] + [\mathcal{C}_{\pi'}] \ ,$$

There are now four cases depending on whether p or p' is free or satellite. We only consider the case when p is free and p' is satellite, the other three cases being similar. Write $p \in E$ and $\{p'\} = E \cap E'$ for exceptional divisors E, E' with Farey weights (a, b), (a', b'), respectively. Then

$$\begin{aligned}[\mathcal{C}_{\pi\circ\pi'}] &= \pi'^*(a-1)[E] + [E'] \\ &= (a-1)[E] + a[E'] = (a-1)[E] + (a'-1)[E']\end{aligned}$$

as $a' = a + 1$ in this case. This concludes the proof. ☐

Proof of Lemma C.3. We use Puiseux series as in Section B. Let (a, b) be the Farey weight of E, $\nu = \Phi(E) \in \mathcal{V}$ the associated divisorial valuation and π the contraction map, respectively. In local source coordinates at a generic point on $E = (x = 0)$ and target coordinates at the origin we have $\pi(x, y) = (x^c\star, x^d\star)$, where $\star$ denotes elements of $R \setminus \mathfrak{m}$. By permuting the two target coordinate axes if necessary, we can assume that $c \le d$. Hence $c = \min\{\pi_*\mathrm{div}_E(x), \pi_*\mathrm{div}_E(y)\} = b\min\{\nu(x), \nu(y)\} = b$. After changing x slightly we obtain $\pi(x, y) = (x^b, x^d\star)$. By Lemma C.7, the Jacobian determinant of the contraction map π is given by $x^{a-1}\star$, which becomes x^{a-1} after changing the source coordinates slightly. Thus $d = a - b$ and π is of the form

$$\pi(x, y) = (x^b, x^d(y + \sum_{i\ge 0} a_i x^{c_i}))$$

with $a_i \ne 0$, $c_{i+1} > c_i \ge c_0 = 0$. We have $\pi_*\mathrm{div}_E = b\,\nu$ by Lemma C.5. Notice that $\nu(x) = 1 \le d/b = -1 + a/b = \nu(y)$. Thus ν extends to a valuation on $\bar{k}[y]$ of the form $\nu = \mathrm{val}\,[\hat\phi; \hat\beta]$ for a Puiseux series $\hat\phi \in \hat{k}$, and where $1 + \hat\beta \in \mathbf{Q}_+$ is the thinness of ν. See Theorem B.6. On the other hand, $\hat\beta$ is also the maximum of $\nu(y - \hat\psi)$ when $\hat\psi$ ranges over all Puiseux series. But

$$\nu(y - \hat\psi) = b^{-1}\pi_*\mathrm{div}_E(y - \hat\psi) = b^{-1}\mathrm{div}_E\left(yx^d + \sum a_i x^{c_i+d} - \hat\psi(x^b)\right) \le d/b,$$

with equality when $\hat\psi(x^b) \in \mathfrak{m}^d$. Hence $\hat\beta = d/b$ and $A(\nu) = 1 + \hat\beta = a/b$. ☐

Proof of Lemma C.4. Let us fix some notation and terminology. A vertex $E \in \Gamma_{\mathbf{Q}}$ can be viewed as the last exceptional divisor of a blowup π at a finite sequence

$(p_i)_1^n$ of infinitely nearby points above the origin. Any finite truncation $(p_i)_0^k$ defines a blowup π_k with last exceptional divisor E_k. The point p_{k+1} is thus a free (satellite) point depending if it belongs to the regular (singular) set of $\pi_k^{-1}(0)$.

We order the free points: set $n_1 = 1$ and, inductively, n_k the minimal integer greater than n_{k-1} such that p_{n_k} is free. Set $\bar{E}_k = E_{n_k-1}$. The dual graph Γ_π of π then looks as in Figure 6. See [Sp] for more information. We let $\nu_k = \Phi(E_k)$ be the

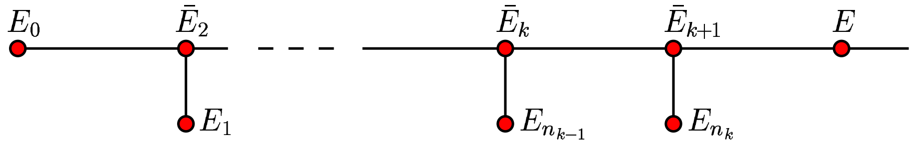

FIGURE 6. The dual graph

divisorial valuation associated to E_k and write $\bar{\nu}_k = \nu_{n_k-1}$. Proving Lemma C.4 now amounts to showing:

(i) $\nu_{n_k} \geq \overline{\nu}_k$.
(ii) $\overline{\nu}_k \leq \nu_j \leq \nu_{n_k}$ for $n_k \leq j \leq n_{k+1} - 1$.
(iii) $\nu_{n_k} \wedge \nu_{n_{k-1}} = \overline{\nu}_k$.

Notice that (i) is a consequence of Lemma C.6. As for (ii) and (iii) we consider the restriction of $\pi = \pi_{n_k-1}$ to a neighborhood of p_{n_k}. The critical set of π has one irreducible component E_{n_k-1}. Choose coordinates centered at p_{n_k} so that $E_{n_k-1} = \{x = 0\}$.

Consider the monomial valuations μ_s defined by $\mu_s(x) = 1$, $\mu_s(y) = s$ for $s \leq 1$. Their associated sequences of infinitely nearby points contain only satellite points: these are located at the singular set of the total transform of E_{n_k-1}. Conversely any valuation centered at p_{n_k} whose associated sequence of infinitely nearby points consists of only satellite points is of the form μ_s (see Example 1.12).

In particular, this is the case for the lift of ν_j to p_{n_k} for $n_k \leq j \leq n_{k+1} - 1$, hence ν_j is proportional to $\pi_*\mu_{s_j}$ for some $s_j \leq 1$. Note that $\mu_0 = \mathrm{div}_{E_{n_k-1}}$, hence $\pi_*\mu_0 = C\,\nu_{n_k-1}$ for some $C > 0$. As p_{n_k} is a free point, the proof of Lemma C.6 implies that the valuation $C^{-1}\pi_*\mu_s$ is normalized for any s. In particular, $C^{-1}\,\pi_*\mu_{s_j} = \nu_j$ for $n_k \leq j \leq n_{k+1} - 1$. As $C^{-1}\,\pi_*\mu_1 = \nu_{n_k}$ and $\mu_s \leq \mu_1$ for $s \leq 1$, we conclude that $\nu_j \leq \nu_{n_k}$. This proves (ii).

To prove (iii) we note that the image under π of the curve $\{y = 0\}$ is an irreducible curve $\{\phi = 0\}$ whose total transform is given by $\pi^*\phi = x^m y$ for some $m \geq 1$. The argument above shows that $\nu_{n_k}(\phi) = C^{-1}(1 + m)$ and $\overline{\nu}_{k+1}(\phi) = \nu_{n_{k+1}-1}(\phi) = C^{-1}(s + m)$ for some $s < 1$. Thus $\overline{\nu}_{k+1}(\phi) < \nu_{n_k}(\phi)$. On the other hand, the strict transform of $\phi = 0$ (i.e. of $y = 0$) does not contain $p_{n_{k+1}}$, hence does not pass through $p_{n_{k+1}}$. Thus $\overline{\nu}_{k+1}(\phi) = \nu_{n_{k+1}}(\phi)$. This proves that ν_{n_k} and $\nu_{n_{k+1}}$ do not define the same tree tangent vector at $\overline{\nu}_{k+1}$. Hence (iii) holds. This completes the proof of Lemma C.4. ☐

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