ON CONSTRUCTION OF SATURATED DISTINGUISHED CHAINS *

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Abstract. Let v be a Henselian valuation of arbitrary rank of a field K and $\bar{v}$ be the (unique) extension of v to a fixed algebraic closure $\overline{K}$ of K. For an element α belonging to $\overline{K}\backslash K$, a chain $\alpha = \alpha_0, \alpha_1, ..., \alpha_r$ of elements of $\overline{K}$ such that α_i is of minimum degree over K with the property that $\bar{v}(\alpha_{i-1} - \alpha_i) = \sup\{\bar{v}(\alpha_{i-1} - \beta) \mid [K(\beta) : K] < [K(\alpha_{i-1}) : K]\}$ and that $\alpha_r \in K$, is called a saturated distinguished chain for α with respect to (K, v). The notion of a saturated distinguished chain has been used to obtain results about the irreducible polynomials over any complete discrete rank one valued field K and to determine various arithmetic and metric invariants associated to elements of $\overline{K}$ (cf. [*J. Number Theory,* 52 (1995), 98-118.] and [*J. Algebra,* 266 (2003), 14-26]). In this paper, we describe a method of constructing a saturated distinguished chain for α and also determine explicitly some invariants associated to α, when the degree of the extension $K(\alpha)/K$ is not divisible by the characteristic of the residue field of v.

Key words: Non-archimedean valued fields; Valued fields.

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§1. *Introduction.* Throughout v is a Henselian valuation of arbitrary rank of a field K with value group G and $\bar{v}$ is (unique) prolongation of v to a fixed algebraic closure $\overline{K}$ of K with value group $\overline{G}$. By the degree of an element α belonging to $\overline{K}$, we shall mean the degree of the extension $K(\alpha)/K$ and denote it by deg α. A finite extension $(K', v')/(K, v)$ (or briefly K'/K) will be called defectless if $[K' : K] = ef$, where e, f are respectively the index of ramification and the residual degree of v'/v. For α belonging to $\overline{K}\backslash K$, let $\delta_K(\alpha)$ denote the supremum of the set $M(\alpha, K)$ defined by

$$M(\alpha, K) = \{\bar{v}(\alpha - \beta) |\ \beta \in \overline{K},\ [K(\beta) : K] < [K(\alpha) : K]\},$$

where for the definition of supremum, the value group of $\bar{v}$ may be viewed as a subset of its Dedekind order completion defined in the usual way. In 2002, Khanduja, Popescu and Roggenkamp proved that $M(\alpha,\ K)$ has an upper bound in $\overline{G}$ if and only if $[K(\alpha) : K] = [\tilde{K}(\alpha) : \tilde{K}]$, where $(\tilde{K},\ \tilde{v})$ is completion of $(K,\ v)$ (see [4,Theorem 3.1]). Later it was proved that $M(\alpha,\ K)$ has a maximum element for every $\alpha \in \overline{K}\backslash K$ if and only if each simple extension of (K, v) is defectless (see [1]). If a Henselian valued field (K, v) has the above property, then to each α belonging to $\overline{K}\backslash K$, one can associate an element α_1 of smallest degree over K such that $\bar{v}(\alpha - \alpha_1) = \delta_K(\alpha)$. Such an ordered pair (α, α_1) is referred to as a distinguished pair. In other words, a pair (α, α_1) of elements of the algebraic closure of a Henselian valued field (K, v) is called a distinguished pair (more precisely (K, v)-distinguished pair) if the following three conditions are satisfied: a) $\bar{v}(\alpha - \alpha_1) = \delta_K(\alpha)$, b) $\deg \alpha > \deg \alpha_1$, c) if γ belonging to $\overline{K}$ has degree less than that of α, then $\bar{v}(\alpha - \gamma) < \bar{v}(\alpha - \alpha_1)$.

Distinguished pairs give rise to distinguished chains in a natural manner. A chain $\alpha = \alpha_0, \alpha_1, ..., \alpha_r$ of elements of $\overline{K}$ will be referred to as a saturated distinguished chain for α if (α_i, α_{i+1}) is a distinguished pair for $0 \leq i \leq r-1$ and $\alpha_r \in K$.

Popescu and Zaharescu [6] were the first to introduce the notion of distinguished pairs, distinguished chains and some invariants associated to these chains. It is known that the length r of a saturated distinguished chain $\alpha = \alpha_0, \alpha_1, ..., \alpha_r$ for an element α together with the degrees $[K(\alpha_i) : K]$ and the values $\bar{v}(\alpha_{i-1} - \alpha_i)$, $1 \leq i \leq r$, are invariants associated to α (see [2, Theorems 1.4, 1.5], [6, Proposition 4.3]). In this paper, taking (K, v) to be a Henselian valued field of arbitrary rank, our aim is to give a procedure for constructing explicitly a saturated distinguished chain for α and to determine the above mentioned invariants associated to such a chain for α when the degree of the extension $K(\alpha)/K$ is not divisible by the characteristic of the residue field of v. Indeed we prove:

THEOREM 1.1. *Suppose that (K, v) is a Henselian valued field of arbitrary rank and α is algebraic over K of degree n not divisible by the characteristic of the residue field of v. Let $c_1 > c_2 > ... > c_r$ be all the distinct members of the set $\{\bar{v}(\alpha - \alpha') \mid \alpha' \neq \alpha$ runs over K-conjugates of $\alpha\}$ and t_i be the number of elements in the set $\{\alpha' \mid \alpha'$runs over K-conjugates of α with $\bar{v}(\alpha - \alpha') = c_i\}$. For $1 \leq j \leq r$, let S_j be the set of those K-conjugates α' of α for which $\bar{v}(\alpha - \alpha') \geq c_j$. If α_j is defined by*

$$\alpha_j = \frac{1}{\mid S_j \mid} \sum_{\alpha' \in S_j} \alpha' = (t_1 + ... + t_j + 1)^{-1} \sum_{\alpha' \in S_j} \alpha',$$

then $\alpha = \alpha_0, \alpha_1, ..., \alpha_r$ *is a saturated distinguished chain for* α, *deg* $\alpha_j = n/(t_1 + ... + t_j + 1)$ *and* $\bar{v}(\alpha_{j-1} - \alpha_j) = c_j$ *for* $1 \leq j \leq r$.

It may be pointed out that the above theorem generalizes Theorem 3.2 in [5] which is proved in the special case when (K, v) is a local field, i.e., a finite extension of the field of p-adic numbers or of $F_p((T))$. Our proof of this theorem is more or less self-contained.

§2. *Notations and preliminary results.* Let (K, v) and $(\overline{K}, \bar{v})$ be as in the opening lines of the paper. As usual $Tr_{K'/K}$ will stand for the trace map from a finite extension K' of K. For $\alpha \in K^{sep} \backslash K$, we shall denote by $\omega_K(\alpha)$, the Krasner's constant defined by

$$\omega_K(\alpha) = \max\{\bar{v}(\alpha - \alpha') \mid \alpha' \neq \alpha \text{ runs over } K\text{-conjugates of } \alpha\}.$$

The following well known lemma will be used in the sequel (cf. [3, 16.8]).

KRASNER'S LEMMA. *Let* (K, v) *and* $(\overline{K}, \bar{v})$ *be as above and* α *be an element of* $K^{sep} \backslash K$. *If* β *belonging to* $\overline{K}$ *is such that* $\bar{v}(\alpha - \beta) > \omega_K(\alpha)$, *then* $K(\alpha) \subseteq K(\beta)$.

We prove a lemma which will be used in the proof of Theorem 1.1. It is of independent interest as well.

LEMMA 2.1. *Let* (K, v) *be a Henselian valued field of arbitrary rank and* $(\overline{K}, \bar{v})$ *be as above. Suppose that* α *is an element of* $\overline{K} \backslash K$, *whose degree is not divisible by the characteristic of the residue field of* v. *Let* K' *be a finite Galois extension of* K *containing* α *and* M *be the fixed field of the group* $\{\sigma \in Gal(K'/K) \mid \bar{v}(\alpha - \sigma(\alpha)) \geq \omega_K(\alpha)\}$. *Suppose that* β *belonging to* M *is* *such that $\bar{v}(\alpha-\beta)=\omega_K(\alpha)$. Then (α,β) is a distinguished pair and $M=K(\beta)$.*

Proof. Note that the group $H=\{\sigma\in \mathrm{Gal}(K'/K)\mid \bar{v}(\alpha-\sigma(\alpha))\geq\omega_K(\alpha)\}$ is a proper subgroup of $\mathrm{Gal}(K'/K(\alpha))$; consequently the fixed field M of H is properly contained in $K(\alpha)$. In particular deg $\beta<$ deg α. So by the definition of $\delta_K(\alpha)$, we have $\bar{v}(\alpha-\beta)\leq\delta_K(\alpha)$. Using the hypothesis that $\bar{v}(\alpha-\beta)=\omega_K(\alpha)$, we see that

$$\delta_K(\alpha)\geq\bar{v}(\alpha-\beta)=\omega_K(\alpha).$$

The above inequality must be an equality, for otherwise there would exist γ belonging to $\overline{K}$ such that $\deg\gamma<\deg\alpha$ and $\bar{v}(\alpha-\gamma)>\omega_K(\alpha)$, which in view of Krasner's Lemma [3, 16.8] would imply $K(\alpha)\subseteq K(\gamma)$ leading to a contradiction. Hence $\delta_K(\alpha)=\omega_K(\alpha)$. So to prove that (α,β) is a distinguished pair, it is enough to prove that whenever θ belonging to $\overline{K}$ is such that

$$\bar{v}(\alpha-\theta)=\omega_K(\alpha), \tag{1}$$

then we must have

$$\deg\theta\geq\deg\beta. \tag{2}$$

Let θ be as in (1). If $\beta\in K$, then (2) is trivially true. It may be assumed that β does not belong to K. The desired inequality (2) will follow once we show that

$$\omega_K(\alpha)>\omega_K(\beta), \tag{3}$$

because (1), (2) and (3) will then give

$$\bar{v}(\beta-\theta)\geq\min\{\bar{v}(\beta-\alpha),\bar{v}(\alpha-\theta)\}=\omega_K(\alpha)>\omega_K(\beta)$$

which by virtue of Krasner's Lemma will imply $K(\beta)\subseteq K(\theta)$ thereby establishing (2). We now verify (3). Observe that if a member τ of $\mathrm{Gal}(K'/K)$ is such that $\tau(\beta) \neq \beta$, then the fact $K(\beta) \subseteq M$ implies that τ does not belong to $\mathrm{Gal}(K'/M) = H$ and hence by the definition of H, we have

$$\bar{v}(\alpha - \tau(\alpha)) < \omega_K(\alpha). \tag{4}$$

With τ as above, it follows from (4) and the strong triangle law that

$$\bar{v}(\beta - \tau(\beta)) = \bar{v}(\beta - \alpha + \alpha - \tau(\alpha) + \tau(\alpha) - \tau(\beta))$$

$$= \min\{\bar{v}(\alpha - \tau(\alpha)), \omega_K(\alpha)\} = \bar{v}(\alpha - \tau(\alpha))$$

which shows that

$$\bar{v}(\beta - \tau(\beta)) = \bar{v}(\alpha - \tau(\alpha)) < \omega_K(\alpha).$$

As the above inequality holds for all those $\tau \in \mathrm{Gal}(K'/K)$ with $\tau(\beta) \neq \beta$, we see that $\omega_K(\beta) < \omega_K(\alpha)$ which proves (3) and completes the proof of the assertion that (α, β) is a distinguished pair.

We now show that $M = K(\beta)$. As $K(\beta) \subseteq M$, we only need to verify that $\mathrm{Gal}(K'/K(\beta)) \subseteq \mathrm{Gal}(K'/M) = H$. If $\sigma \in \mathrm{Gal}(K'/K(\beta))$, then keeping in view the hypothesis $\bar{v}(\alpha - \beta) = \omega_K(\alpha)$, we see that

$$\bar{v}(\alpha - \sigma(\alpha)) = \bar{v}(\alpha - \beta + \sigma(\beta) - \sigma(\alpha))$$

$$\geq \min\{\bar{v}(\alpha - \beta),\ \bar{v} \circ \sigma(\beta - \alpha)\} = \omega_K(\alpha)$$

which shows that $\sigma \in H$. Therefore $M = K(\beta)$.

§3. *Proof of Theorem 1.1.* Let K' be the smallest normal extension of K containing α. For $1 \leq i \leq r$, let H_i denote the subgroup of $\mathrm{Gal}(K'/K)$ defined by $H_i = \{\sigma \in \mathrm{Gal}(K'/K) \mid \bar{v}(\alpha - \sigma(\alpha)) \geq c_i\}$ and M_i be the fixed field of H_i. Then $t_1 + ... + t_i + 1$ being the degree of the extension $K(\alpha)/M_i$ is a factor of

$n = [K(\alpha) : K]$ and hence it is not divisible by the characteristic of the residue field of v by virtue of the hypothesis. In particular $t_1 + ... + t_i + 1 \neq 0$. If α_i is as in the theorem, then it can be easily seen that

$$\alpha_i = \frac{1}{t_1 + ... + t_i + 1} Tr_{K(\alpha)/M_i}(\alpha). \tag{5}$$

For convenience of notation, we shall denote the degree $t_1 + 1$ of the extension $K(\alpha)/M_1$ by m. We first show that (α, α_1) is a distinguished pair with α_1 given by

$$\alpha_1 = \frac{1}{m} \sum_{\alpha' \in S_1} \alpha' = \frac{1}{m} Tr_{K(\alpha)/M_1}(\alpha).$$

As $\alpha_1 \in M_1$, we have

$$[K(\alpha_1) : K] \leq \frac{n}{m} < [K(\alpha) : K]. \tag{6}$$

If $\alpha^{(1)}, ..., \alpha^{(m)}$ are the elements of S_1, then keeping in mind that $v(m) = 0$, we see that

$$\bar{v}(\alpha - \alpha_1) = \bar{v}(m\alpha - m\alpha_1) = \bar{v}(m\alpha - \sum_{i=1}^{m} \alpha^{(i)}) \geq \min_i \bar{v}(\alpha - \alpha^{(i)}) = \omega_K(\alpha). \tag{7}$$

In view of Krasner's Lemma and (6), the inequality in (7) implies that

$$\bar{v}(\alpha - \alpha_1) = \omega_K(\alpha). \tag{8}$$

Applying Lemma 2.1, we see that (α, α_1) is a distinguished pair and

$$M_1 = K(\alpha_1); \tag{9}$$

in particular, deg $\alpha_1 = n/m = n/(t_1 + 1)$. We prove the theorem by induction on the length r of a saturated distinguished chain for α. It is already proved when $r = 1$ in the above paragraph keeping in mind that $\omega_K(\alpha)$ is denoted by c_1. Before applying induction hypothesis on α_1, we show that

$$\{\alpha_1' \mid \alpha_1' \neq \alpha_1 \text{ runs over } K\text{-conjugates of } \alpha_1\} = \{c_2, ..., c_r\}. \tag{10}$$

To verify (10), note that by virtue of (9), for an automorphism τ of K'/K, $\tau(\alpha_1) \neq \alpha_1$ if and only if τ does not belong to $\mathrm{Gal}(K'/M_1) = H_1$ which holds if and only if $\bar{v}(\alpha - \tau(\alpha)) < \omega_K(\alpha)$; for such an automorphism τ keeping in view the equality $\bar{v}(\alpha - \alpha_1) = \omega_K(\alpha)$ established in (8), we have by the strong triangle law

$$\bar{v}(\alpha_1 - \tau(\alpha_1)) = \min\{\bar{v}(\alpha_1 - \alpha), \bar{v}(\alpha - \tau(\alpha)), \bar{v}(\tau(\alpha) - \tau(\alpha_1))\} = \bar{v}(\alpha - \tau(\alpha)).$$

This proves (10).

For $1 \leq i \leq r-1$, let $c_i^{'}$ stand for c_{i+1} and t_i' denote the cardinality of the set $\{\alpha_1' \mid \alpha_1^{'}$ runs over K-conjugates of α_1 with $\bar{v}(\alpha_1 - \alpha_1^{'}) = c_i'\}$. Let $H_i^{'}$ denote the group $\{\sigma \in \mathrm{Gal}(K'/K) \mid \bar{v}(\alpha_1 - \sigma(\alpha_1)) \geq c_i^{'}\}$ and M_i' the fixed field of H_i'. Arguing as for the proof of (10) in above paragraph, we see that $H_i^{'} = H_{i+1}$; consequently

$$M_i^{'} = M_{i+1}, \quad 1 \leq i \leq r-1. \tag{11}$$

Since $K(\alpha)$ is a separable extension of $M_1 = K(\alpha_1)$, every K-isomorphism σ defined on $K(\alpha_1)$ can be extended to $K(\alpha)$ in exactly $m = t_1 + 1$ ways, therefore the argument cited above also shows that

$$mt_i^{'} = t_{i+1}, \; 1 \leq i \leq r-1. \tag{12}$$

Applying induction hypothesis to α_1 and using (12) and (10), we see that $\alpha_1 = \beta_o, \beta_1, ..., \beta_{r-1}$ is a saturated distinguished chain for α_1, where

$$\beta_{j-1} = \frac{1}{1 + t_1^{'} + ... + t_{j-1}^{'}} Tr_{K(\alpha_1)/M_{j-1}'}(\alpha_1),$$

$$\deg \beta_{j-1} = \deg \alpha_1/(1 + t_1^{'} + ... + t_{j-1}^{'}) = \deg \alpha/(1 + t_1 + ... + t_j)$$

and

$$\bar{v}(\beta_{j-1} - \beta_j) = c_{j+1}, \; 1 \leq j \leq r-1.$$

It only remains to be verified that $\beta_{j-1} = \alpha_j$ for $2 \leq j \leq r$. Recall that as in (5) we can write $\alpha_1 = \frac{1}{t_1+1} Tr_{K(\alpha)/M_1}(\alpha)$. Using (11) and (12), it is clear that

$$\beta_{j-1} = \frac{1}{1+t_1+...+t_j} Tr_{K(\alpha_1)/M_j}(Tr_{K(\alpha)/M_1}(\alpha)).$$

Consequently keeping in mind (9), we have

$$\beta_{j-1} = \frac{1}{1+t_1+...+t_j} Tr_{M_1/M_j}(Tr_{K(\alpha)/M_1}(\alpha)) = \frac{1}{1+t_1+...+t_j} Tr_{K(\alpha)/M_j}(\alpha).$$

Therefore $\beta_{j-1} = \alpha_j$ by virtue of (5). This completes the proof of the theorem.

As an application of the above theorem, we give an example.

EXAMPLE. Let $\mathbb{Q}$ be the field of rational numbers and p_1, p_2, p_3 be distinct rational primes. Let $K = \mathbb{Q}((t))$ be the field of Laurent series in an indeterminate t and v denote the t-adic valuation on K with $v(t) = 1$. Let $\overline{K}$ and $\bar{v}$ have the same meaning as above. Consider $\alpha = t + t\sqrt{p_1} + t^2\sqrt{p_2} + t^3\sqrt{p_3}$. Keeping in mind that any automorphism of $\overline{K}/K$ maps $\sqrt{p_i}$ to $\pm\sqrt{p_i}$, it is clear that the set

$$\{\bar{v}(\alpha - \alpha') \mid \alpha' \neq \alpha \text{ runs over } K\text{-conjugates of } \alpha\} = \{1, 2, 3\}.$$

So the saturated distinguished chain for α will have length 3. With the notations of Theorem 1.1, $c_1 = 3, c_2 = 2, c_3 = 1$ and the sets

$$S_i = \{\alpha' \mid \ \bar{v}(\alpha - \alpha') \geq c_i\}$$

are given as follows:

$S_1 = \{\alpha,\ t + t\sqrt{p_1} + t^2\sqrt{p_2} - t^3\sqrt{p_3}\}$.

$S_2 = S_1 \bigcup\ \{t + t\sqrt{p_1} - t^2\sqrt{p_2} + t^3\sqrt{p_3},\ t + t\sqrt{p_1} - t^2\sqrt{p_2} - t^3\sqrt{p_3}\}$.

$S_3 = S_2 \bigcup\ \{t - t\sqrt{p_1} + t^2\sqrt{p_2} + t^3\sqrt{p_3},\ t - t\sqrt{p_1} + t^2\sqrt{p_2} - t^3\sqrt{p_3},\ t - t\sqrt{p_1} - t^2\sqrt{p_2} +\ t^3\sqrt{p_3},\ t - t\sqrt{p_1} - t^2\sqrt{p_2} - t^3\sqrt{p_3}\}$.

Therefore by Theorem 1.1, $\alpha, \alpha_1, \alpha_2, \alpha_3$ is a saturated distinguished chain for α with

$$\alpha_1 = \frac{1}{\mid S_1 \mid} \sum_{\alpha' \in S_1} \alpha' = t + t\sqrt{p_1} + t^2\sqrt{p_2},$$

$$\alpha_2 = \frac{1}{\mid S_2 \mid} \sum_{\alpha' \in S_2} \alpha' = t + t\sqrt{p_1}, \quad \text{and} \quad \alpha_3 = \frac{1}{\mid S_3 \mid} \sum_{\alpha' \in S_3} \alpha' = t.$$

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