1 Definition and basic constructions

This first section presents part of Rosenlicht’s paper [R01]. In the ring \( \mathbb{R}^\mathbb{R} \) of all functions from \( \mathbb{R} \) to \( \mathbb{R} \) (with pointwise addition and multiplication of functions), consider the ideal

\[ \mathcal{N} := \{ f \in \mathbb{R}^\mathbb{R} \mid \exists x_0 \in \mathbb{R} : f|_{(x_0, \infty)} = 0 \} \]

Then \( \mathbb{R}^\mathbb{R}/\mathcal{N} \) is a reduced \( \mathbb{R} \)-algebra, whose elements can be considered as germs of functions at infinity. Usually, we write \( f + \mathcal{N} \in \mathbb{R}^\mathbb{R}/\mathcal{N} \) simply as \( f \). Writing \( f \leq g \) if \( f(x) \leq g(x) \) for all sufficiently large \( x \in \mathbb{R} \) defines a partial order on \( \mathbb{R}^\mathbb{R}/\mathcal{N} \) which is compatible with the ring operations and antisymmetric, that is, \( f \leq g \leq f \iff f = g \). The identity function (germ) is denoted by \( x \) rather than \( \text{id}_\mathbb{R} \). Also note that the non-units in \( \mathbb{R}^\mathbb{R}/\mathcal{N} \) are precisely the zero divisors, and they are characterised as the functions which have arbitrarily large zeroes.

If \( R \subseteq \mathbb{R}^\mathbb{R} \) is a subring, then, by abuse of notation, we write \( R/\mathcal{N} \) to mean the subring \( R/R \cap \mathcal{N} \subseteq \mathbb{R}^\mathbb{R}/\mathcal{N} \). For \( n \in \mathbb{N}_0 \cup \{ \infty \} \), let \( C^n := C^n(\mathbb{R}, \mathbb{R}) \subseteq \mathbb{R}^\mathbb{R} \) be the ring of \( n \) times continuously differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \), also called \( C^n \)-functions. Then, since any \( C^n \)-function defined on some interval \( (x_0, \infty) \) has a \( C^n \)-continuation to all of \( \mathbb{R} \), the ring \( \mathcal{C}_n := C^n/\mathcal{N} \) can be considered as the ring of germs of \( C^n \)-functions at infinity. We set \( \mathcal{C} := \bigcap_n \mathcal{C}_n \). Note that \( \mathcal{C}_n^* = \mathcal{C}_0^* \cap \mathcal{C}_n \) for all \( n \in \mathbb{N}_0 \cup \{ \infty \} \), and \( \mathcal{C}^* = \mathcal{C}_0^* \cap \mathcal{C} \).

1.1. Warning. \( \mathcal{C}_\infty \subseteq \mathcal{C} \).

Proof. Let \( f : \mathbb{R} \to \mathbb{R} \) be given by

\[ f(x) := \begin{cases} 0 & \text{for } x \leq 0, \\ 1 + \int_0^{x-1} f(t)dt & \text{for } x > 0. \end{cases} \]
Then \( f'(x) = f(x - 1) \) for \( x > 1 \), and so \( f^{(n)}(x) = f(x - n) \) for \( x > n \) by induction on \( n \in \mathbb{N}_0 \). Thus, for all \( n \in \mathbb{N}_0 \), \( f^{(n)} \) is continuous on \( (n, \infty) \) but not differentiable at \( x = n + 1 \). Therefore \( f \in \mathcal{C} \setminus \mathcal{C}_0 \).

\[ \Box \]

1.2. Lemma. Any subfield \( K \subseteq \mathcal{C}_0 \) is totally ordered.

Proof. Let \( f \in K^* \). Then \( f \in \mathcal{C}_0^* \). By the Intermediate Value Theorem, it follows that \( f > 0 \) or \( f < 0 \).

Note that derivation

\[
\begin{align*}
\mathcal{C}_n & \to \mathcal{C}_{n-1} \\
f & \mapsto f'
\end{align*}
\]

is a well-defined \( \mathbb{R} \)-linear map. A **Hardy field** is a subfield \( K \subseteq \mathcal{C}_1 \) which is closed under \( f \mapsto f' \). Note that this implies \( \mathbb{Q} \subseteq K \subseteq \mathcal{C} \), and a Hardy field is an ordered, differential field. The intersection of any collection of Hardy fields is obviously again a Hardy field. As we will see later, there is no such statement for composites of Hardy fields.

1.3. Example. Let \( k \) be any subfield of \( \mathbb{R} \). Then \( k, k(x), k(\exp), k(x, \log) \) are Hardy fields, but \( k(x^2), k(\log), \mathbb{Q}(\exp(\sqrt{2}x)), (\mathbb{Q} \cap \mathbb{R})(2^x) \) are not Hardy fields because they are not closed under derivation. Since \( \sin, \cos \notin \mathcal{C}_0^* \), they are contained in no Hardy field.

We call \( f \in \mathcal{C} \) a **Hardy function** or **D-consistent** (see next section) if \( f \) is contained in some Hardy field. Then, in particular, \( f' \in \mathcal{C}^* \cup \{0\} \) so that \( f(\infty) := \lim_{x \to \infty} f(x) \in \mathbb{R} \cup \{\pm \infty\} \) is well-defined.

1.4. Problem. Is \( f \in \mathcal{C} \) as defined in 1.1 a Hardy function?

In order to construct (or assure the existence of) certain Hardy fields, we establish two theorems concerning algebraic and first order differential adjunction, respectively. We shall make use of the Implicit Function Theorem in the following form.

**Implicit Function Theorem.** Let \( U \subseteq \mathbb{R}^n \) and \( V \subseteq \mathbb{R}^m \) be open, \( F : U \times V \to \mathbb{R}^m \), \((x, y) \mapsto F(x, y)\) a \( C^1 \)-function and \( a \in U \), \( b \in V \) such that \( F(a, b) = 0 \) and \( \frac{\partial}{\partial y} F(a, b) \in \text{GL}_m(\mathbb{R}) \). Then there are open subsets \( U' \subseteq U \) and \( V' \subseteq V \) with \( a \in U' \), \( b \in V' \) and a \( C^1 \)-function \( g : U' \to V' \) such that for all \( x \in U' \), \( y \in V' \), we have \( F(x, y) = 0 \) if and only if \( y = g(x) \). Moreover, 

\[
g'(a) = -\left( \frac{\partial}{\partial y} F(a, b) \right)^{-1} \cdot \frac{\partial}{\partial x} F(a, b).
\]
1.5. Theorem. Let $y \in C_0$ be algebraic over the Hardy field $K$. Then $K[y]$ is a Hardy field.

Proof. Let $Y$ be a variable and $F \in K[Y]$ the minimum polynomial of $y$ over $K$. We aim at showing that $F$ is irreducible in $K[Y]$. Assume $F = G \cdot H$ in $K[Y]$. Then $AG + BH = 1$ for some $A, B \in K[Y]$. Choose $x_0 \in \mathbb{R}$ large enough so that $F(y) = 0$ and $A(y)G(y) + B(y)H(y) = 1$ as functions on $(x_0, \infty)$. Then, on this half line, $G(y)$ and $H(y)$ must have disjoint and closed zero sets, whose union is the connected set $(x_0, \infty)$ because $G(y)H(y) = F(y) = 0$. Therefore $G(y) = 0$ or $H(y) = 0$ in $C_0$, showing that $F$ is a power of an irreducible. But since $K[Y]/(F) \simeq K[y] \subseteq C_0$ is reduced, $F$ must itself be irreducible, and $K[y] = K(y)$ is a field.

Write $F = \sum_k a_k Y^k$ with $a_k \in K$. Since $F(y) = 0$, and $F$ and $F' = \sum k a_k Y^{k-1}$ are coprime, $F'(y)$ is invertible in $K[y] = K(y)$. So the Implicit Function Theorem yields $y \in C_1$ and $y' \in K(y)$. Now let $G = \sum_k g_k Y^k \in K[Y]$ be arbitrary. Then $G(y') = G'(y) y' + \sum_k g_k y^k \in K[y, y'] = K(y)$ showing that $K(y)$ is a Hardy field. $\square$

As a consequence we obtain

1.6. Theorem. Let $K \subseteq C$ be a Hardy field. Then

$$
\bar{K} := \{ f \in C_0 \mid f \text{ is algebraic over } K \}
$$

is a real closed Hardy field.

Proof. Clearly, $\bar{K}$ is a Hardy field since for any $f, g \in \bar{K}$, by Theorem 1.5, $K[f]$ and $K[f, g]$ are Hardy fields. It remains to be shown that any (monic) irreducible $F \in K[Y]$ of degree $n \in \mathbb{N}$ has $n$ distinct roots $g_1, \ldots, g_n \in K(i)$, where $i = \sqrt{-1}$. Write $F = \sum_{k=0}^n a_k Y^k$. Since $F$ and $F'$ are coprime and $a_k \in K$, we can choose $x_0 \in \mathbb{R}$ large enough so that all $a_k$ are $C^1$ on $(x_0, \infty)$ and so that, for all $x > x_0$,

$$
F_x := F(x, Y) = \sum_{k=0}^n a_k(x) Y^k \in \mathbb{R}[Y]
$$

has $n$ distinct roots in $\mathbb{C}$. Consider $F$ as a function $(x_0, \infty) \times \mathbb{C} \to \mathbb{C}$, $(x, z) \mapsto F(x, z) = F_x(z)$ and identify $C = \mathbb{R}^2$ on setting $i = (0, 1)$. For fixed $x > x_0$, being a polynomial, $F_x$ is holomorphic, say with complex derivative

3
For each root $w$ of $F_x$, we have $F'_x(w) = \alpha + i\beta \neq 0$ with certain $\alpha, \beta \in \mathbb{R}$, so that $\frac{\partial}{\partial z} F(x, w) = \left( \frac{\alpha - \beta}{\alpha} \right) \in \text{GL}_2(\mathbb{R})$ by the Cauchy-Riemann differential equations. By the Implicit Function Theorem, there exist an open subinterval $I \subseteq (x_0, \infty)$ and $C^1$-functions $g_1, \ldots, g_n : I \to \mathbb{C}$ such that for any $x \in I$, $g_1(x), \ldots, g_n(x)$ are the $n$ distinct roots of $F_x$. Since we could enlarge $I$ by applying the same argument to its end points, the maximal choice of such an interval is $I = (x_0, \infty)$. Now write $g_k = u_k + iv_k$ with $C^1$-functions $u_k, v_k : I \to \mathbb{R}$. Then, for each $k$ there exists $l$ with $u_k - iv_k = u_l + iv_l$, so that both $u_k$ and $v_k$ are algebraic over $K$, showing that $g_k \in \overline{K}(i)$. 

The second construction concerns solutions of certain first order differential equations. Its proof uses the

**Picard Lindelöf Theorem.** Let $G \subseteq \mathbb{R} \times \mathbb{R}^m$ be open, $(a, b) \in G$ and $F : G \to \mathbb{R}^m$ a continuous function which is $C^1$ in the second variable. Then there is $\varepsilon > 0$ and a unique $C^1$-function $g : I = [a - \varepsilon, a + \varepsilon] \to \mathbb{R}^m$ such that $g(a) = b$ and $g'(x) = F(x, g(x))$ for all $x \in I$.

1.7. **Theorem.** Let $K$ be a Hardy field, $y \in C_1$ and $G, H \in K[Y]$ such that $H(y) \in C_0^*$ and $Y = G(y)/H(y) \in C_0$. Then $K[y]$ is an integral domain, and its field of fractions $K(y)$ is a Hardy field.

**Proof.** It suffices to show that $K[y] \subseteq C_0^* \cup \{0\}$. For this, we may assume that $K = \overline{K}$ is real closed. We prove by contradiction. Suppose there is a polynomial $F \in K[Y]$ with

$$0 \neq F(y) \in C_0 \setminus C_0^*.$$ 

Because then at least one of the (monic) irreducible factors of $F$ satisfies (1), we may assume that $F$ is itself monic irreducible. Hence $F$ is either linear or quadratic with negative discriminant. But in the second case we would have $F(y) > 0$, violating (1), so that $F = Y - a$ for some $a \in K$. Replacing $y$ by $y - a$, we can assume that $0 \neq y \in C_1 \setminus C_0^*$.

Then $H(y) \in C_0^*$ implies that $Y$ does not divide $H$ in $K[Y]$. If $Y | G$, then the differential equation $y' = G(z)/H(z)$ would have the two solutions $z = 0$ and $z = y \neq 0$, which coincide for arbitrarily large values of $z$, contradicting the uniqueness part of the Picard Lindelöf Theorem. Therefore $Y$ does not divide $GH$, that is, $G = \sum a_k Y^k$, $H = \sum b_k Y^k$ with $a_k, b_k \in K$ and $a_0 b_0 \neq 0$. Choose $x_0 \in \mathbb{R}$ large enough so that $a_0$ and $b_0$ have constant sign and $y' = G(y)/H(y)$ on $(x_0, \infty)$. Since $0 \neq y \in C_1 \setminus C_0^*$, there is $x_2 > x_0$.
such that \( y(x_2) \neq 0 \) but \( y(x) = 0 \) for some \( x \in (x_0, x_2) \), and we can set \( x_1 := \sup\{x_0 < x < x_2 \mid y(x) = 0\} \) and \( x_3 := \inf\{x_2 < x \mid y(x) = 0\} \). Then \( x_0 < x_1 < x_2 < x_3 \) and \( y(x) \neq 0 \) for all \( x \in (x_1, x_3) \). Also, since \( y \) is continuous, \( y(x_1) = 0 = y(x_3) \) and \( y \) has constant sign on \( (x_1, x_3) \). By the other assumptions, \( y'(x_1) = a_0(x_1)/b_0(x_1) \) and \( y'(x_3) = a_0(x_3)/b_0(x_3) \) have the same sign, a contradiction. \( \square \)

1.8. Example. Let \( k \) be any subfield of \( \mathbb{R} \). Then \( k(x, \sqrt{\log(x)}, \exp(\sqrt{\log(x)})) \), \( k(x, \text{arctan}) \), \( k(x, \exp(1 + 1/x)) \) and \( k(y, y') = k(y') \), where \( y' = \sqrt{2 + 1/y} \) with \( y(0) = 1 \), are Hardy fields.

Combining Theorems 1.6 and 1.7 yields the following

1.9. Corollary. Let \( K \) be a maximal Hardy field. Then \( K \) is real closed and contains every \( y \in C_1 \) that satisfies a differential equation \( y' = F(y) \) for some \( F \in K(Y) \). In particular, \( \mathbb{R}(x) \subseteq K \), and for each \( f \in K^* \), all primitives of \( f \) are in \( K \) and \( \exp(f, \log|f|) \), \( \text{arctan} f \), \( |f|^r \in K \) for any \( r \in \mathbb{R} \).

Remark. By Zorn's Lemma, every Hardy field is contained in a maximal Hardy field, which thus satisfies all the aforementioned properties. The existence of such an extension will be used frequently without further mention.

1.10. Corollary. If \( K \) is a Hardy field and \( k \) is some subfield of \( \mathbb{R} \), then the compositum \( kK = K(k) \subseteq C \) is again a Hardy field.

1.11. Example. \( \mathbb{R}(x^r \mid r \in \mathbb{R}) \) and \( \mathbb{R}(x^r, \exp(ax^r) \mid a, r \in \mathbb{R}) \) are Hardy fields.

1.12. Corollary. Let \( K \) be a maximal Hardy field and \( a, b \in K \). Suppose that

\[
(2) \quad y'' + ay' + by = 0
\]

for some \( y \in C_2 \). Then \( K \) contains all solutions to (2).

Proof. By assumption \( v := y'/y \in C_1 \) satisfies the Riccati equation \( v' + v^2 + av + b = 0 \). Hence \( v \in K \) and, consequently, \( y \in K \) by 1.9. Let \( u \in K \) be a non-constant solution to \( u'' + (a + 2v)u' = 0 \). Then \( uy \) also solves (2) and is \( \mathbb{R} \)-linearly independent of \( y \). \( \square \)

5
2 Boshernitzan’s field $E$ and the existence of transexponential functions

Throughout this section, $K$ is a Hardy field. A subset $S \subseteq C$ is called $D$-consistent if $S$ is contained in some Hardy field, and in that case we define $E[S]$ to be the intersection of all maximal Hardy fields containing $S$, which obviously is again a Hardy field. In particular, we will consider the Boshernitzan field

$$E := E[0] = E[Q] = E[\mathbb{R}(x)].$$

As will be seen later, there are uncountably many maximal Hardy fields, so $E$ is far from being maximal. Still, $E$ is much larger than Hardy’s original field $L$ of logarithmico-exponential functions (see [Ha1, Ha2]), in that $L \subseteq E$ is not even a Liouville extension (while $\mathbb{R} \subseteq L$ is). The field $E$ has been defined and studied extensively by Boshernitzan [Bo1, Bo2].

A subset $S \subseteq C$ will be called $D$-consistent with $K$ if $K \cup S$ is $D$-consistent, and then $K(S)$ denotes the smallest Hardy field containing both $K$ and $S$. A $D$-consistent subset $S \subseteq C$ is automatically $D$-consistent with $E$, and then $E(S) \subseteq E[S]$. Moreover, the elements of $E$ are precisely those functions $f \in C$ which are $D$-consistent with all Hardy fields.

Note that $f \in C$ is $D$-consistent with $K$ if and only if $K[f, f', f'', \ldots] \subseteq C^* \cup \{0\}$, and then

$$K(f) = K(f, f', f'', \ldots).$$

We observe that the elements of $E$ are precisely those functions $f \in C$ which are $D$-consistent with all Hardy fields. Furthermore, $S \subseteq C$ is $D$-consistent iff $S$ is $D$-consistent with $E$, and then $E(S) \subseteq E[S]$.

By definition, $E$ shares all the properties of maximal Hardy fields mentioned in Corollary 1.9. Unlike the maximal Hardy fields, $E$ has the additional property of being closed under composition of function germs. To show this, we need some preparations. Let $h \in C_0$ and $f \in C_1$ with $f(\infty) = \infty$ and $f' > 0$. Then the composition $h \circ f \in C_0$ and the compositional inverse $g = f_{-1} \in C_1$ are well-defined, $g(\infty) = \infty$, $g' = 1/f' \circ g > 0$ and $g \circ f = f \circ g = x$. Moreover, $K \circ f = \{h \circ f \mid h \in K\}$ and $K \circ g \subseteq C_1$ are fields.

2.1. Lemma. Let $K \not\subseteq \mathbb{R}$ be a Hardy field and $f \in C_1$ with $f(\infty) = \infty$ and $f' > 0$. Then $f' \in K$ if and only if $K \circ f_{-1}$ is a Hardy field.
Proof. Set \( g = f_{-1} \). First assume \( f' \in K \). Then
\[
(h \circ g)' = (h' \circ g) \cdot g' = h' \circ g \in K \circ g
\]
for any \( h \in K \), showing that \( K \circ g \) is closed under differentiation. For the converse, choose a non-constant \( h \in K \). Then \( h' \circ g = (h \circ g)' \in K \circ g \) implies \( h' \in K \), and \( h' \neq 0 \), so that \( f' \in K \). \qed

2.2. Corollary. If \( f \in C_0 \) is \( D \)-consistent and \( f(\infty) = \infty \), then \( f_{-1} \) is \( D \)-constistant, too.

Proof. Clearly, \( x, x' \in K := E(f) \), so \( f_{-1} \in K \circ f_{-1} \) is \( D \)-consistent by the previous lemma. \qed

2.3. Lemma. Let \( M \) be a maximal Hardy field and \( f \in M \) with \( f(\infty) = \infty \). Then \( M \circ f_{-1} \) is a maximal Hardy field.

Proof. By 2.1, \( M \circ f_{-1} \) is a Hardy field. To prove it is maximal, let \( M' \) be a Hardy field containing \( M \circ f_{-1} \). Since \( x \in M \), we have \( f_{-1} \in M \circ f_{-1} \subseteq M' \), so \( M' \circ f \) is a Hardy field by 2.1. Then \( M = M \circ f_{-1} \circ f \subseteq M' \circ f \). Moreover, \( M \subseteq M' \circ f \) hence \( M = M' \circ f \) by the maximality of \( M \), that is \( M \circ f_{-1} = M' \). \qed

2.4. Theorem. The field \( E \) is closed under composition, that is, for \( f, g \in E \) with \( f(\infty) = \infty \), also \( g \circ f \in E \).

Proof. Let \( f, g \in E \) and \( f(\infty) = \infty \). Then for every maximal Hardy field \( M \) we have \( g \in E \subseteq M \circ f_{-1} \) by definition of \( E \) and the previous lemma. Hence \( g \circ f \in E \) by definition of \( E \). \qed

We say that \( f \in C \) satisfies an **algebraic differential equation (ADE) over** \( K \) of order \( \leq n \in \mathbb{N}_0 \) if \( F(f, f', \ldots, f^{(n)}) = 0 \) for some polynomial \( F \in K[Y_0, Y_1, \ldots, Y_n] \).

2.5. Remark. Let \( f \in C \) be \( D \)-consistent with the Hardy field \( K \), and suppose \( f \) satisfies an ADE over \( K \) of order \( \leq n \in \mathbb{N}_0 \). Then \( K(f) = K(f, f', \ldots, f^{(n)}) \).

2.6. Theorem. Let \( f \in E \). Then \( f \) satisfies an ADE over \( \mathbb{R} \), and \( f \) is (real) analytic on some positive half line.
The $n$-th compositional iterate of a function $f \in C_0$ with $f(\infty) = \infty$ will be denoted by $f_n$. We say that a function $f \in C_0$ is (a) transponential if it is larger than any compositional iterate of exp, that is, $f > \exp_n$ for all $n \in \mathbb{N}$; in particular, $f(\infty) = \infty$. A Hardy field not containing a transponential is called exponentially bounded. By 1.9, every function in such a Hardy field is bounded by some iterate of exp.

2.7. Remark. Every $f \in C_0$ satisfying the functional equation $f(x + 1) = e^{f(x)}$ is transponential.

2.8. Proposition. There is $f \in C_\infty$ with $f(x + 1) = e^{f(x)}$ and $f' > 0$.

Proof. Define $f$ in a small neighbourhood $[-\epsilon, \epsilon]$ of 0 by a $C^\infty$-function with $f' > 0$, large enough so that its values are strictly less than those of $e^{f(x-1)}$ on $[1 - \epsilon, 1 + \epsilon]$. Connect the two parts so that we get a $C^\infty$-function $f$ on $[-\epsilon, 1 + \epsilon]$ with $f' > 0$. Finally, we extend the definition to the whole positive half line using the functional equation.

Our goal is to show that the transexponentials $f$ in the previous proposition are $D$-consistent, that is, $\mathbb{R}[f, f', f'', \ldots] \subseteq C_* \cup \{0\}$. We start with some simple observations.

2.9. Lemma. Let $g \in C_0$, $h \in C_*^*$ satisfy $|g(x) - g(x - 1)| \leq h(x) \leq \frac{1}{3} h(x + 1)$. Then $|g| < 2h$.

Proof. We may assume that both functions are continuous and that the double inequality holds numerically for all $x > x_0 \in \mathbb{R}$. Write $x = x_1 + m$ with $m \in \mathbb{N}_0$ and $x_1 \in (x_0 - 1, x_0]$, and set $c = \max_{x \in [x_0 - 1, x_0]} |g(x)| \in \mathbb{R}$. Then

$$|g(x)| \leq |g(x_1)| + \sum_{k=1}^{m} h(x_1 + k) \leq c + \sum_{k=1}^{m} 3^{k-m} h(x_1 + m) \leq c + \frac{3}{2} h(x).$$

Since $h(\infty) = \infty$ by assumption, we conclude $|g| < 2h$. 

2.10. Lemma. Let $g \in C_n$, $h \in C_n^*$. Then

$$(g/h)^{(n)} = F(g, g', \ldots, g^{(n)}, h, h', \ldots, h^{(n)})/h^{n+1}$$

for some polynomial $F \in \mathbb{Z}[X_0, \ldots, X_n, Y_0, \ldots, Y_n]$ which is homogeneous of degree 1 in $X_0, \ldots, X_n$ and homogeneous of degree $n$ in $Y_0, \ldots, Y_n$. 

\textbf{Proof.} This is easily seen by induction on \( n \). \( \square \)

We will also use logarithmic derivatives. Therefore we set \( \partial u = u'/u \) for \( u \in C_1^* \) and \( \partial^n u = \partial \cdots \partial u \) whenever this is defined.

\textbf{2.11. Lemma.} Let \( n \in \mathbb{N}_0 \) and \( f \in C_n \) such that \( \partial^n f \) is defined. Then \( f^{(n)} \in \mathbb{Z}[f, \ldots, \partial^n f] \).

\textit{Proof.} We proceed by induction on \( n \). The case \( n = 0 \) is trivial. For the induction step assume that \( f^{(n)} = F(f, \partial f, \ldots, \partial^n f) \) for some polynomial \( F \in \mathbb{Z}[X_1, \ldots, X_n] \). Writing \( (\partial^n f)' = (\partial^n f)(\partial^{n+1} f) \), we see that \( f^{(n+1)} = F(f, \partial f, \ldots, \partial^n f)' \in \mathbb{Z}[f, \partial f, \ldots, \partial^{n+1} f] \). \( \square \)

In this context we also consider the ideal \( a := \{ a \in C \mid \forall n \in \mathbb{N}_0 : a^{(n)} = 0 \} \) of the ring \( \{ f \in C \mid \forall n \in \mathbb{N}_0 : f^{(n)}(\infty) \in \mathbb{R} \} \).

\textbf{2.12. Lemma.} \( 1 + a = \{ 1 + a \mid a \in a \} \) is a subgroup of \( C^* \) and \( \partial : 1 + a \to a \) is an injective group homomorphism.

\textit{Proof.} Let \( u, v \in 1 + a \). Then \( (u/v)(\infty) = 1 \) and \( u'v - uv' \in a \), so \( (u/v)^{(n)}(\infty) = 0 \) for all \( n \in \mathbb{N} \) by Lemma 2.10. Hence \( u/v \in 1 + a \), and \( 1 + a \) is a subgroup of \( C^* \). By the ideal property of \( a \), we have \( u'/u \in a \). So \( \partial : 1 + a \to a \) is well-defined, and a group homomorphism because \( \partial(uv) = \partial u + \partial v \). To see that \( \partial \) is injective on \( 1 + a \), assume \( \partial u = 0 \), that is \( \log |u| = c \in \mathbb{R} \). Then \( |u| = e^c \) is constant, too, and \( u(\infty) = 1 \) forces \( u = 1 \). (Note that the map is not onto because \( \partial x = 1/x \in a \) but \( cx \notin a \) for any \( c \in \mathbb{R}^* \).) \( \square \)

\textbf{2.13. Proposition.} Let \( f \in C \) satisfy \( f(x + 1) = e^{f(x)} \) and \( f' > 0 \). Then

(a) \( f(x) = f(x)f'(x - 1) \).

(b) \( f < f' < f^2 \), and \( \partial f(\infty) = \infty \).

(c) \( |f^{(n)}| < (\log f)^n f \) for all \( n \in \mathbb{N} \).

(d) \( \partial(f'u)/f'(x - 1) \in 1 + a \) for any \( u \in 1 + a \).

(e) \( \partial^n f \) is well-defined and satisfies \( \partial^n f(x)/f'(x - n) \in 1 + a \) for all \( n \in \mathbb{N} \).

(f) \( \partial^n f > (\partial^{n+1} f)^m \) for all \( m, n \in \mathbb{N}_0 \).

\textit{Proof.} (a) Differentiating the functional equation yields \( f'(x + 1) = e^{f'(x)}f'(x) = f(x + 1)f'(x) \).
(b) Since $0 < f' \in C_0$ and $f(\infty) = \infty$, there are $x_0, \varepsilon > 0$ such that $f' > \varepsilon$ on $[x_0, x_0 + 1]$ and $f > 1$ on $(x_0, \infty)$. By (a) we conclude that $\partial f = f'(x)/f(x) = f'(x - 1)$ tends to $\infty$. In particular, $f < f'$. Again by (a), $g = \log f'$ and $h = \log f$ satisfy the requirements of Lemma 2.9. Hence $f' < f^2$ as asserted.

(c) We proceed by induction on $n$. For $n = 1$, note that $|f'| = f'(x) = f(x)f'(x - 1) < f(x)(f(x - 1))^2 = f(x)(\log f(x))^2$. To show the assertion for $n + 1$, choose $x_0$ large enough, set

$$c := \max_{0 \leq \nu \leq n+1} \max_{x \in [x_0, x_0 - 1]} |f^{(\nu)}(x)|$$

and write $x > x_0$ as $x = x_1 + m$ with $m \in \mathbb{N}$ and $x_1 \in (x_0 - 1, x_0]$. Using (a) repeatedly $m$ times we obtain $f(x) = f'(x_1) \prod_{k=0}^{m-1} f(x - k)$. Differentiating this $n$ times yields

$$f^{(n+1)}(x) = \sum_{\nu \in \mathbb{N}} \binom{n}{\nu} P_{\nu}(x) f^{(\nu + 1)}(x_1)$$

where $\binom{n}{\nu} = \frac{m!}{\nu! \cdot (m - \nu)!}$ and $P_{\nu}(x) = \prod_{k=0}^{m-1} f^{(\nu_k)}(x - k)$ for $\nu \in \mathbb{N} = \{(\nu_0, \ldots, \nu_m) \in \mathbb{N}^{m+1} | \nu_0 + \cdots + \nu_m = n\}$. By induction hypothesis we have

$$|P_{\nu}(x)| \leq \prod_{k=0}^{m-1} (f(x - k)(f(x - k - 1))^{2\nu_k})$$

$$\leq f(x)(f(x - 1))^{2(\nu_0 + \cdots + \nu_{m-1})} \prod_{k=1}^{m-1} f(x - k)$$

$$\leq f(x)(f(x - 1))^{2n+3/2},$$

hence

$$|f^{(n+1)}(x)| \leq \sum_{\nu \in \mathbb{N}} \binom{n}{\nu} |P_{\nu}(x) f^{(\nu + 1)}(x_1)|$$

$$\leq (m + 1)^n c f(x) (f(x - 1))^{2n+3/2}$$

$$\leq c(x - x_0 + 2)^n (\log f(x))^{2n+3/2} f(x),$$

and we can conclude $|f^{(n+1)}| < (\log f)^{2n+2} f$. 
(d) Set \( \phi = f''/f' \) and let \( u \in 1 + a \). Applying \( \partial \) to (a) we get \( \phi(x) = \partial f'(x) = f'(x)/f(x) + \partial f'(x - 1) = f'(x - 1) + \phi(x - 1), \) hence

\[
\partial(f'u) = f'(x - 1)(1 + \frac{\phi}{f'}(x - 1) + \frac{\partial u(x)}{f'(x-1)})
\]

and it remains to show that both fractions in the brackets lie in \( a \). Differentiating the former equation \( n \in \mathbb{N}_0 \) times yields

\[
|\phi^{(n)}(x) - \phi(n)(x - 1)| = |f^{(n+1)}(x - 1)| < (\log f(x - 1))^{2n+2} f(x - 1)
\]

by (c), so that Lemma 2.9 with \( g = \phi^{(n)} \) and \( h \) being equal to the right hand side of the previous inequality gives \( |\phi^{(n)}| < 2h < (\log f)^2 \). Together with Lemma 2.10 and (c) we conclude that for all \( n \in \mathbb{N}_0 \) there is \( c > 0 \) such that

\[
|\phi/f'|(n) \leq c(\log f)^2((\log f)^{2n+2} f)/(f')^{n+1} \leq (f/f')^{n+1}.
\]

So \( \phi/f' \in a \) by (b). Similarly, since \( \partial u \in a \), for all \( n \in \mathbb{N}_0 \) there is \( d > 0 \) such that

\[
(\partial u(x+1)/f'(x))^{(n)} \leq d((\log f)^{2n+2} f)/(f')^{n+1} \leq (f/f')^{n+1}.
\]

(e) is proven by induction on \( n \in \mathbb{N} \). The equality \( \partial f(x) = f'(x)/f(x) = f'(x - 1) \) follows from (a). For the induction step assume that \( \partial^{n+1} f(x) = f'(x - n)u_n(x) \) with some \( u_n \in 1 + a \). Then

\[
\partial^{n+1} f(x) = \partial(f'(x - n)u_n(x)) = f'(x - n - 1)u_{n+1}(x)
\]

for some \( u_{n+1} \in 1 + a \) by (d).

(f) By (a) and (c) for any \( \varepsilon > 0 \) we have

\[
\partial f(x) = f'(x - 1) \leq (\log f(x - 1))^2 f(x - 1) \leq (\log f(x))^2 \leq (f(x))^{\varepsilon}.
\]

This settles the assertion for \( n = 0 \). Using (e), (a) and (b), we obtain

\[
2\partial^n f(x) > f'(x - n) > f(x - n) = e^{f(x-n-1)} > e\sqrt{f(x-n-1)} > e\sqrt{\partial^{n+1} f(x)/2} > 2(\partial^{n+1} f(x))^m
\]

for all \( m, n \in \mathbb{N} \). \qed
Now the $D$-consistency of $f$ is an easy consequence of 2.13(f) and Lemma 2.11. In fact, we can prove even more.

2.14. Theorem. Let $K$ be an exponentially bounded Hardy field and $f \in \mathcal{C}$ as in the previous proposition. Then each of the two families $(\partial^n f)_{n \in \mathbb{N}_0}$ and $(f^{(n)})_{n \in \mathbb{N}_0}$ is algebraically independent over $K$. Moreover, $f$ is $D$-consistent with $K$.

Proof. Let $0 \neq F \in K[X_0, \ldots, X_n]$, equip $\mathbb{N}_0^{n+1}$ with the lexicographical order, write $F = \sum_{\nu \in \mathbb{N}_0^{n+1}} a_\nu X^\nu$, and take $\nu \in \mathbb{N}_0^{n+1}$ minimal with $a_\nu \neq 0$. Then by 2.13(g) we have

$$F(f, \partial f, \ldots, \partial^n f)(\infty) = \begin{cases} \infty & \text{if } a_\nu > 0 \\ -\infty & \text{if } a_\nu < 0, \end{cases}$$

showing that $f, \partial f, \ldots, \partial^n f$ are $K$-algebraically independent. By Lemma 2.11, we conclude $F(f, f', \ldots, f^{(n)}) \in K[f, \partial f, \ldots, \partial^n f] \subseteq C^* \cup \{0\}$. Hence, $K(f, f', \ldots, f^{(n)}) = K(f, \partial f, \ldots, \partial^n f) \subseteq \mathcal{C}$ has transcendence degree $n + 1$ over $K$, and $f$ is $D$-consistent with $K$. \hfill \Box

2.15. Corollary. Every maximal Hardy field contains a transexponential.

The smallest Hardy field containing a transexponential such as in Proposition 2.13 is $\mathbb{Q}(f) = \mathbb{Q}(f, \partial f, \partial^2 f, \ldots)$. This Hardy field does not contain $\exp_n$ for any $n \in \mathbb{Z}$ since, by 2.13(f) and the reasoning in the previous proof, every $g \in \mathbb{Q}(f)$ with $g(\infty) = \infty$ is transexponential.

2.16. Theorem. There are uncountably many maximal Hardy fields. In particular, $E$ is not maximal.

Proof. Let $f \in \mathcal{C}$ satisfy $f(x + 1) = e^{f(x)}$ and $f' > 0$. For $-1 < 2\pi r < 1$ we define

$$f_r(x) = f(x + r \sin(2\pi x)).$$

Then $f_r \in \mathcal{C}$, $f_r(x + 1) = f(x + r \sin(2\pi x) + 1) = e^{f_r(x)}$ and $f'_r(x) = f'(x + r \sin(2\pi x))(1 + 2\pi r \cos(2\pi x)) > 0$, so that $f_r$ is $D$-consistent by the previous theorem. Now let $-1 < 2\pi r < 2\pi s < 1$. For every large enough $x_0 \in \mathbb{R}$ there are $x_1, x_2 > x_0$ with $\sin(2\pi x_1) > 0 > \sin(2\pi x_2)$, hence

$$f_r(x_1) < f_s(x_1) \text{ and } f_r(x_2) > f_s(x_2).$$

Therefore $f_r$ and $f_s$ cannot lie in the same Hardy field. Since every $f_r$ for $0 < 2\pi r < 1$ lies in some maximal Hardy field, there are uncountably many of them. \hfill \Box
3 The rank

This section is an exposition of part of Rosenlicht’s paper [Ro2]. We start by briefly recalling the notion of archimedean equivalence and natural valuation. Let \((G, +, 0, \leq)\) be an ordered abelian group and \(a, b \in G\). We set

\[
|a| := \begin{cases} 
a & \text{if } a \geq 0 
-b & \text{if } a \leq 0
\end{cases}
\]

and write \(a \geq b\) if \(n|a| \geq |b|\) for some \(n \in \mathbb{N}\). This induces an equivalence relation on \(G\), whose classes are denoted

\[
[a] = \{g \in G \mid a \leq g \leq a\}
\]

and are called the archimedean classes of \(G\). We agree that \(\infty := [0] = \{0\}\) and

\([a] \leq [b] \iff a \geq b\)

to obtain the chain \([G] = \{[g] \mid g \in G\}\) with greatest element \(\infty\). It is easily verified that

\([a + b] \geq \min\{[a], [b]\}\).

The rank of \(G\) is defined as (the order type of) \([G] \setminus \{\infty\}\) and denoted rank \(G\). If \(G\) is the additive group of an ordered field \((K, +, \cdot, 0, 1, \leq)\), then

\([a] + [b] := [ab]\)

is a well-defined operation, turning \(vK := [K] \setminus \{\infty\}\) into an ordered abelian group with identity element \(0 := [1]\). Hence the map

\[v: K \to [K] = vK \cup \{\infty\}, \quad g \mapsto [g] = vg\]

is a (Krull) valuation on \(K\), called the natural valuation of the ordered field \(K\). By definition, \(v\) has valuation ring \(\mathcal{O} = \{g \in K \mid \exists n \in \mathbb{N} : |g| \leq n\}\) with maximal ideal \(m = \{g \in K \mid \forall n \in \mathbb{N} : n |g| \leq 1\}\). Hence, its residue field \(K_v := \mathcal{O}/m\) is naturally contained in \(\mathbb{R}\) as an ordered field.
We call two elements \( f, g \in K^* \) (multiplicatively) comparable if \([vg] = [vf]\). Note that \( \pm f, \pm 1/f \) are mutually comparable. Obviously, comparability is an equivalence relation on \( K^* \), giving \( v^{-1}[vf] \subseteq K^* \) as the comparability class of \( f \). In particular, we have the trivial class \( v^{-1}\{0\} = \mathcal{O}^* \). The non-trivial comparability classes form a chain isomorphic to \([vK] \setminus \{\infty\}\). This chain, or rather its order type, is defined to be the rank of the ordered field \( K \). Hence \( \text{rank } K := \text{rank}(vK) \).

**Remark.** Let \( K \) be a Hardy field. Then \( vf \) can be thought of as the order of vanishing at infinity of \( f \in K \). The place associated with \( v \) is given by

\[
K \rightarrow Kv \cup \{\infty\} \\
f \rightarrow f_v := \begin{cases} 
\infty & \text{if } f(\infty) = \pm \infty, \\
 f(\infty) & \text{otherwise.}
\end{cases}
\]

In particular, \( Kv \) naturally contains the constant field \( K \cap \mathbb{R} \).

The lexicographic product of two ordered abelian groups \( G \) and \( H \) is denoted \( G \times H \). Similarly, if \( \Delta \) is some chain, the group \( G(\Delta) \) of all functions from \( \Delta \) to \( G \) with finite support is agreed to be ordered lexicographically from the left. Note that then \( \text{rank } G(\Delta) \simeq \Delta \).

### 3.1. Example.

Let \( k \) be any subfield of \( \mathbb{R} \).

(a) The Hardy field \( K = k(x) \) has value group \( vK \simeq \mathbb{Z} \), and \( v \) is just the degree valuation, that is, \( vg = -\deg g \) for all \( g \in K \). Hence, \( \text{rank } K = 1 \).

(b) If \( K = k(x, \exp) \), then \( vK \simeq \mathbb{Z} \times \mathbb{Z} \) with \( v(x^{-\alpha} \exp^{-\beta}) = (\beta, \alpha) \). Hence \( \text{rank } K = 2 \).

(c) Let \( \ell_n \) for \( n \in \mathbb{Z} \) denote the \( n \)-th compositional iterate of \( \log \), that is, \( \ell_2 = \log \circ \log \), \( \ell_1 = \log \), \( \ell_0 = x \), \( \ell_{-1} = \exp \), \( \ell_{-2} = \exp \circ \exp \), \ldots. Then the Hardy field \( K = k(\ell_n \mid n \in \mathbb{Z}) \) has value group \( vK = \mathbb{Z}(\mathbb{Z}) \), that is, \( v\prod_{n} \ell_n^{-\alpha_n} = (\alpha_n)_{n \in \mathbb{Z}} \) is positive iff \( m = \min \{n \in \mathbb{Z} \mid \alpha_n \neq 0 \} \) exists and \( \alpha_m > 0 \). In particular, \( \text{rank } K \simeq \mathbb{Z} \).

For the rest of this section, \( K \) will always denote a Hardy field, \( v \) its natural valuation and \( \mathcal{O} \) its valuation ring. Our first aim is to show that \( \text{rank } K \) can be naturally embedded into \( vK \) as a chain. First we collect some facts about how \( v \) behaves together with (logarithmic) derivatives.
3.2. Lemma. Let \( f, g \in K^* \).
(a) Suppose \( vf \neq 0 \neq vg \). Then \( vf \leq vg \iff vf' \leq vg' \).
(b) If \( 0 \neq vf < vg \), then \( vf' < vg' \).
(c) If \( vf \neq 0 \leq vg \), then \( v(f'/f) < vg' \).
(d) If \( 0 < |vg| \leq |vf| \), then \( v(f'/f) \leq v(g'/g) \).
(e) Suppose \( f(\infty) = g(\infty) = \infty \). Then
\[
[vf] \leq [vg] \iff \exists n \in \mathbb{N} : f^n > g \iff \exists n \in \mathbb{N} : f^n \geq g.
\]
(f) Suppose \( vf \neq 0 \neq vg \). Then \( [vf] \leq [vg] \iff v(f'/f) \leq v(g'/g) \).

\textit{Proof.} (a) By assumption, we have \( f(\infty), g(\infty) \in \{0, \pm \infty\} \) and \( f' \neq 0 \neq g' \), so that \( \frac{f}{g}(\infty) = \frac{g'}{f'}(\infty) \) by l'Hospital's rule. Hence \( vf \leq vg \iff \frac{f}{g}(\infty) \in \mathbb{R} \iff \frac{g'}{f'}(\infty) \in \mathbb{R} \iff vf' \leq vg' \).
(b) follows from (a) if \( vg \neq 0 \). Otherwise, \( v(g-c) > vg = 0 \) for some \( c \in \mathbb{R} \), so again \( vf' < v(g-c') = vg' \).
(c) Since \( f(\infty) \in \{0, \pm \infty\} \), we have \( v(\log |f|) < 0 \leq vg \), so the result follows from (b).
(d) After replacing \( f \) by \( 1/f \) and \( g \) by \( 1/g \) if necessary, we can assume \( vf \leq vg < 0 \). Then \( N |f| \geq |g| \) for some \( N \in \mathbb{N} \), and so \( \frac{\log |f|}{\log |g|}(\infty) \geq 1 \). We conclude \( v(\log |f|) \leq v(\log |g|) \ < 0 \), whence the result follows by (a).
(e) \( [vf] \leq [vg] \iff \exists n \in \mathbb{N} : -nvf = n |vf| \geq |vg| = -vg \iff \exists n \in \mathbb{N} : v f^n \leq vg \iff \exists n, N \in \mathbb{N} : N f^n \geq g \iff f^{n+1} > g \iff f^{n+1} \geq g. \)
(f) We may assume \( f(\infty) = g(\infty) = \infty \), hence also \( \log f(\infty) = \log g(\infty) = \infty \). Using (e) and (a), we obtain \( [vf] \leq [vg] \iff \exists n \in \mathbb{N} : f^n \geq g \iff \frac{\log f(\infty)}{\log g(\infty)} \in \mathbb{R} \iff v(\log f) \leq v(\log g) \iff v(f'/f) \leq v(g'/g). \)

3.3. Corollary. For any \( f \in K^* \), \( vf \geq 0 \) implies \( vf' > 0 \). In particular, \( vK = 0 \) means \( K \subseteq \mathbb{R} \).

\textit{Proof.} From \( vx < 0 \leq vf \) we conclude \( 0 < v(1/x) < vf' \) using 3.2(c).

By Lemma 3.2, the map
\[
\psi_K : vK \setminus \{0\} \rightarrow vK
\]
\[
\psi_K(vf) = v(f'/f)
\]
is well-defined and its image $\Psi_K := \{ v(f'/f) \mid f \in K^* \backslash \mathcal{O}^* \}$ is bounded from above in $vK$. Moreover, $\psi = \psi_K$ induces an order isomorphism

$$\tilde{\psi} : [vK] \setminus \{ \infty \} \to \Psi_K, [vf] \mapsto v(f'/f),$$

so that we obtain the following

3.4. **Theorem.** The rank of $K$ is order-isomorphic to $\Psi_K$.

3.5. **Example.** Let $k$ be a subfield of $\mathbb{R}$.

(a) For $K = k(x)$ we have $\Psi_K = \{ v^{1/x} \} = \{ 1 \}$.

(b) For $K = k(\exp)$, $\Psi_K = \{ 0 \}$.

(c) Let $K = k(\ell_n \mid n \in \mathbb{Z})$ be as in 3.1(c). Then

$$\Psi_K = \{ v(\ell_n / \ell_{n'}) \mid n \in \mathbb{Z} \} = \left\{ \sum_{m=1}^{n} -e_{-m}, \sum_{m=0}^{n} e_{m} \mid n \in \mathbb{N}_0 \right\},$$

where $e_m = v(1/\ell_m)$ is the $m$-th canonical basis vector in $\mathbb{Z}^{(\mathbb{Z})}$.

The statements of Lemma 3.2 can be reformulated as the following properties of the map $\psi : vK \setminus \{ 0 \} \to vK$.

A1 \hspace{1em} $\psi(m\alpha) = \psi\alpha$

A2 \hspace{1em} $\psi(\alpha + \beta) \geq \min\{ \psi\alpha, \psi\beta \}$

A3 \hspace{1em} $\psi\alpha < |\beta| + \psi\beta$

A4 \hspace{1em} $|\beta| \leq |\alpha| \implies \psi\alpha \leq \psi\beta$

for all $\alpha, \beta \in vK \setminus \{ 0 \}, m \in \mathbb{Z} \setminus \{ 0 \}$. We want to investigate the consequences of these axioms in an abstract setting.

4 \hspace{1em} **Asymptotic couples**

The exposition in this section is based on work by Aschenbrenner and van den Dries [As, AD1, AD2]. An **asymptotic couple** is a pair $(\Gamma, \psi)$, where $(\Gamma, +, 0, \leq)$ is an ordered abelian group and $\psi : \Gamma^* \to \Gamma$ is a map satisfying A1, A2 and A3 for all $\alpha, \beta \in \Gamma^* := \Gamma \setminus \{ 0 \}, m \in \mathbb{Z} \setminus \{ 0 \}$. The image of $\psi$ is denoted by $\Psi := \psi(\Gamma^*)$. For convenience, we extend $\psi$ to $\Gamma_\infty := \Gamma \cup \{ \infty \}$.
by setting $\psi(0) = \psi(\infty) = \infty$. We also define $\Gamma_+ := \{\gamma \in \Gamma | \gamma > 0\}$ and $\Gamma_- := \{\gamma \in \Gamma | \gamma < 0\}$.

An asymptotic couple satisfying also A4 is said to be of $H$-type or is called an $H$-asymptotic couple. Given an asymptotic couple $(\Gamma, \psi)$ and $\gamma \in \Gamma$, note that $(\Gamma, \psi + \gamma)$ with

$$\psi + \gamma : \Gamma^* \to \Gamma, \alpha \mapsto \psi\alpha + \gamma,$$

is again an asymptotic couple. It is of $H$-type if and only if $(\Gamma, \psi)$ is.

4.1. Lemma. Let $(\Gamma, \psi)$ be an asymptotic couple, $\alpha, \beta \in \Gamma^*$ and $n \in \mathbb{N}$. Then

(a) $\psi\alpha > 0 \implies \psi^n\alpha > 0$.
(b) $n\psi\alpha < |\beta| + n\psi\beta$.
(c) $\alpha \neq \beta \implies [\alpha - \beta] < [\psi\alpha - \psi\beta]$.
(d) The map $\text{id} + \psi : \Gamma^* \to \Gamma$, $\gamma \mapsto \gamma + \psi\gamma$ is strictly increasing.

Proof. (a) By A3, $\psi^n\alpha > 0$ implies $\psi^n\alpha < \psi^{n+1}\alpha$. Thus the assertion follows by induction on $n$.

(b) We may assume $\beta > 0$ and $\psi\alpha > \psi\beta$. Replacing $\psi$ by $\psi - \psi\beta$ reduces us to the case $\psi\beta = 0 < \psi\alpha$. Using (a) we conclude $0 < \psi^2\alpha$, hence $\psi(n\psi\alpha - \beta) = \min\{\psi^2\alpha, \psi\beta\} = \psi\beta = 0$ by A2. Therefore A3 yields the implication $n\psi\alpha \geq \beta \implies \psi\alpha < n\psi\alpha - \beta \implies (n-1)\psi\alpha > \beta$, from which we obtain $n\psi\alpha < \beta$ by induction on $n$.

(c) We may assume $\psi\alpha > \psi\beta$, hence $\psi(\alpha - \beta) = \psi\beta$. Then we obtain $n(\psi\alpha - \psi\beta) < |\alpha - \beta| + n\psi(\alpha - \beta) - n\psi\beta = |\alpha - \beta|$ from (b).

(d) Suppose $\alpha < \beta$. Then $\beta - \alpha > \psi\alpha - \psi\beta$ by (c), that is, $\alpha + \psi\alpha < \beta + \psi\beta$.

From now on, let $(\Gamma, \psi)$ be an $H$-asymptotic couple. Then $\bar{\psi} : [\Gamma] \to \Gamma_\infty$, $[\gamma] \mapsto \psi\gamma$ is an order-preserving map of chains. In other words,

$$[\alpha] \leq [\beta] \implies \psi\alpha \leq \psi\beta.$$

for all $\alpha, \beta \in \Gamma$.

4.2. Lemma. Let $\alpha \in \Gamma^*$. If $[\alpha] \geq [\psi\alpha]$, then $[\psi\alpha] = [\psi^2\alpha]$ and $\psi^2\alpha = \psi^3\alpha \in \Gamma^*$.
Proof. By (3), we may assume $|\alpha| > |\psi\alpha|$. Then 4.1(c) yields $|\psi^2\alpha - \psi\alpha| > |\psi\alpha - \alpha| = \min\{|\psi\alpha|, |\alpha|\} = |\psi\alpha|$, hence $|\psi\alpha| = |\psi^2\alpha| \in [\Gamma] \setminus \{\infty\}$ and therefore $\psi^2\alpha = \psi^3\alpha \in \Gamma^*$ by (3).

By 4.1(d) we have

$$\Gamma'_- := (\text{id} + \psi)(\Gamma_-) < \Gamma'_+ := (\text{id} + \psi)(\Gamma_+).$$

The next proposition concerns the image

$$\Gamma' := (\text{id} + \psi)(\Gamma^*) = \Gamma'_- \cup \Gamma'_+$$

of $\text{id} + \psi$.

4.3. Theorem. (a) $\Psi < \Gamma'$, and $\Gamma'$ is a final segment in $\Gamma$.

(b) $\Gamma'_- = \{\gamma \in \Gamma | \gamma < \beta \text{ for some } \beta \in \Psi\}$.

(c) If $\gamma = \sup \Psi \in \Gamma$ exists, then $\Gamma' = \Gamma \setminus \{\gamma\}$ and $\Gamma'_- < \gamma < \Gamma'_+$. Otherwise, $\Gamma' = \Gamma$ and $\Psi \subseteq \Gamma'_-$.

Proof. (a) $\Psi < \Gamma'_+$ holds by A3. As for the second assertion, assume $\alpha \in \Gamma_+$ and $\alpha + \psi\alpha < \gamma \in \Gamma$. Replacing $\psi$ by $\psi - \gamma$, we can assume $\gamma = 0$. Then $0 < \alpha < -\psi\alpha$, hence $|\alpha| > |\psi\alpha|$, so that $\psi^2\alpha = \psi^3\alpha \in \Gamma^*$ by Lemma 4.2. Setting $\beta := -\psi^2\alpha$, we conclude $\beta + \psi\beta = 0 > \alpha + \psi\alpha$, hence $\beta > \alpha > 0$ by 4.1(d), thus showing that $\gamma = 0 \in \Gamma'_+$.

(b) Since $\alpha + \psi\alpha < \psi\alpha$ for $\alpha \in \Gamma_-$, the inclusion ($\subseteq$) is clear. Conversely let $\gamma \in \Gamma$ satisfy $\gamma < \psi\alpha$ for some $\alpha \in \Gamma^*$. We may assume $\gamma = 0 < \alpha$. If $\alpha \leq \psi\alpha$, then $|\alpha| \geq |\psi\alpha|$, hence $0 = \beta + \psi\beta$ with $\beta := -\psi^2\alpha < 0$ by 4.2 and 4.1(a). If $\psi\alpha \leq \alpha$, then $0 < \psi\alpha \leq \psi^2\alpha$ by A4, so $0 = \beta + \psi\beta$ with $\beta = -\psi^3\alpha < 0$. We have shown $\gamma = 0 \in \Gamma'_-$ in both cases.

(c) Note that, by (b), for any $\alpha \in \Gamma$ we have

$$\Gamma'_- < \alpha \iff \Psi \subseteq \alpha.$$

Now suppose there were $\alpha, \beta \in \Gamma$ with $\Gamma'_- < \alpha < \beta < \Gamma'_+$. Then A3 and (4) would imply $\beta < \beta - \alpha + \psi(\beta - \alpha) \leq \beta - \alpha + \alpha = \beta$, a contradiction. Therefore, $|\Gamma \setminus \Gamma'| \leq 1$, and using (4) once more, we also see that $\alpha \in (\Gamma \setminus \Gamma')$ implies $\alpha = \sup \Psi$.

Conversely assume that $\sup \Psi = \gamma \in \Gamma$. Then $\Gamma'_- < \gamma$ by (b). Suppose $\gamma \in \Gamma'_+$, say $\gamma = \alpha + \psi\alpha$ with $\alpha \in \Gamma_+$. Then $\Psi < \gamma = \min \Gamma'_+$ by A3. Hence $\alpha = \min \Gamma'_+$ by 4.1(d). But then $\Psi \subseteq \gamma - \alpha < \gamma = \sup \Psi$, a contradiction. \qed
5 Asymptotic integration

Let us return to the case of a Hardy field $K$ again and continue to exhibit results from [Ro2]. We resume the notation of Section 3 and consider the asymptotic couple $(vK, \psi_K)$ associated with $K$. Note that $(\text{id} + \psi_K)(vg) = vg'$ for $g \in K^* \setminus O^*$. We start by deriving two corollaries from Theorem 4.3.

5.1. Corollary. If $\gamma = \sup \Psi_K \in vK$ exists, then $\gamma \geq 0$. Otherwise $\Psi_K$ contains a strictly positive element.

Proof. We use the notation of the previous section with $\Gamma = vK$. Note that, by Corollary 3.3, we have $0 < \Gamma_+$. If $\gamma = \sup \Psi_K \in vK$ exists, then $\Gamma_+ \subseteq \Gamma' < \gamma$, so $\gamma \geq 0$. Otherwise $0 \in \Gamma \setminus \Gamma'_+ = \Gamma'_-$, hence $0 < \beta$ for some $\beta \in \Psi_K$ according to 4.3(b). \hfill \Box

In view of Example 3.5(b), the preceding corollary seems to be sharp. But it still leaves open one question.

5.2. Problem. Is there a Hardy field $K \not\subseteq \mathbb{R}$ with $\Psi_K < 0$?

By the previous corollary, the rank of such a $K$ cannot have a largest element, and moreover, $vx \not\in vK$ due to Corollary 5.7 below. The second consequence of Theorem 4.3 is the following.

5.3. Corollary. Suppose $g \in K$ with $vg' = \sup \Psi_K$. Then $g(\infty) \in \mathbb{R} \setminus K$. In particular, if $Kv = K \cap \mathbb{R}$, then there is no $g \in K$ with $vg' = \sup \Psi_K$.

Proof. By 4.3(c), $vg = 0$. If we had $c := g(\infty) \in K$, then $v(g - c) > 0$ and $v(g - c)' = vg' = \sup \Psi_K$ contradicting 4.3(c). \hfill \Box

We say that $f, g \in C^*_0$ are **asymptotically equal** and write $f \sim g$ if $f'(\infty) = 1$. Letting $1 + m = \{f \in K^* | f(\infty) = 1\}$ be the group of **one-units**, we note that $K^*/\sim = K^*/(1 + m)$ is an extension of $vK \simeq K^*/O^*$ by $Kv^* \simeq O^*/(1 + m)$, because the sequence

$$1 \to O^*/(1 + m) \to K^*/(1 + m) \to K^*/O^* \to 1$$

is exact.

5.4. Lemma. Let $f, g \in K^*$.

(a) Then $f \sim g \iff v(f - g) > vf$. 

(b) If \(vf = 0\) then \(f \sim f(\infty)\).
(c) Suppose \(f, g \notin \mathbb{R}\) and \(\min\{vf, vg\} \neq 0\). Then \(f \sim g \iff f' \sim g'\).
(d) If \(f \sim g\) then \(\log |f| \sim \log |g|\).

Proof. (a) \(f \sim g \iff \frac{g}{f}(\infty) = 1 \iff v(1 - \frac{g}{f}) > 0 \iff v(f - g) > vf\).
(b) is obvious.
(c) We may assume \(0 \neq vf \leq vg\). Then \(0 \neq vf = v(f - g)\) or \(0 \neq vf = vg\).
In the first case, using (a) and 3.2(a), we obtain the equivalence (of false statements) \(f \sim g \iff v(f - g) > vf \iff v(f' - g') > v f' \iff f' \sim g'\).
In the second case, we have \(f(\infty) = g(\infty) \in \{0, \pm \infty\}\) and \(f' \neq 0 \neq g'\).
Hence \(\frac{f}{g}(\infty) = \frac{f'}{g'}(\infty)\) by l'Hospital’s rule, and the assertion follows.
(d) For \(vf = 0\) this is clear. Otherwise, \(v \log |f| \neq 0 \neq vf = vg\), hence applying (c) twice yields \(f \sim g \implies f'/f \sim g'/g \implies \log |f| \sim \log |g|\). □

As we know from Theorem 1.7, any element \(K\) has an antiderivative in some Hardy extension. We say that \(K\) has asymptotic integration if for every \(f \in K^*\) there is \(g \in K \setminus \mathbb{R}\) with \(f \sim g'\).

5.5. Proposition. Assume that \(K_v = K \cap \mathbb{R}\). Then \(K\) has asymptotic integration if and only if there is no \(\alpha \in vK\) with \(\alpha = \sup \Psi_K\).

Proof. Suppose \(K\) has asymptotic integration, and let \(\alpha \in vK\). Write \(\alpha = vf\) with \(f \in K^*\). Then \(f \sim g'\) for some \(g \in K \setminus \mathbb{R}\). Hence \(\alpha = vf = vg' \neq \sup \Psi_K\) by 5.3.
Conversely, suppose that there is no \(\alpha \in vK\) with \(\alpha = \sup \Psi_K\), and let \(f \in K^*\). Then, by Theorem 4.3, there is \(g \in K^* \setminus \mathcal{O}^*\) with \(vg' = vf\), that is, \(\frac{f}{g'}(\infty) = c \in (K \cap \mathbb{R})^*\), so \(f \sim (cg)'\). □

Even if we drop the assumption on the residues, we can show at least one direction.

5.6. Theorem. Let \(f \in K^*\) with \(vf \neq \sup \Psi_K\). Then there is \(0 < \gamma \in vK\) such that \(g := f \cdot \frac{fa/a'}{(fa/a')^\gamma} \in K^* \setminus \mathcal{O}^*\) and \(f \sim g'\) for all \(a \in K^*\) satisfying \(0 < |va| \leq \gamma\).
Proof. If $\Psi_K \leq v f$, then by assumption there is $f_0 \in K^*$ with $\Psi_K \leq v f_0 < v f$, and we set $\gamma := v(f/f_0)$. Otherwise there is $a_0 \in K^*$ with $va_0 \neq 0$ and $v f < v(a_0/a_0)$, and we set $\gamma := \min\{|va_0|,|v(f a_0/a_0)|\}$. 

Now let $a \in K^*$ with $0 < |va| \leq \gamma$. In the first case, we have $0 < |va| \leq v(f/f_0) \leq v(f a/a') = |v(f a/a')|$ because both $v f$ and $v f_0$ are upper bounds for $\Psi_K$. In the second case, we obtain $v f < v(a_0/a_0) \leq v(a'/a)$ from 3.2(d), hence $0 < |va| \leq |v(f a_0/a_0)| = v(a_0/a_0) - v f \leq v(a'/a) - v f = |v(f a/a')|$. Thus we have shown

$$(5) \quad 0 < |va| \leq |v(f a/a')|. \tag{5}$$

In particular, $v(f a/a') \neq 0$, hence $(fa'/a')' \neq 0$. So $g = f \cdot \frac{fa'/a'}{(fa'/a')'}$ makes sense and has derivative

$$(6) \quad g' = \left(\frac{fa'/a'}{(fa'/a')'}\cdot fa'/a'\right)' = f + \left(\frac{f}{(fa'/a')'}\right)' \cdot fa'.' \tag{6}$$

From (5) and 3.2(d), we conclude $v(a'/a) \geq v(fa'/a')$, that is, $v\frac{f}{(fa'/a')'} \geq 0$. Hence $v(a'/a) < v\left(\frac{f}{(fa'/a')'}\right)'$ by 3.2(c). Together with (6) we obtain

$$v(g' - f) = v f + v\left(\frac{f}{(fa'/a')'}\right)' - v(a'/a) > v f,$$

which, according to 5.4(b), means $g' \sim f$. Finally, if we had $v g = 0$, then $v g' = v f = v\frac{f}{fa'/a'} \in \Psi_K$, contradicting 3.2(c). Hence $g \in K^* \setminus O^*$.

5.7. Corollary. We have the equivalence $\sup \Psi_K \neq 0 \iff \forall x \in v K \iff \exists f \in K^*: f \sim x$.

Proof. From Theorem 5.6, Lemma 5.4(c), Lemma 3.2(a) and Theorem 4.3 we get $\sup \Psi_K \neq 0 \implies \exists f \in K^* \setminus O^*: x' = 1 \sim f' \implies x \sim f \in K^* \implies v x \in v K \implies \exists f \in K: v f = v x < 0, v f' = v x' = 0 \implies \sup \Psi_K \neq 0$.  

From the above theorem we can also derive the existence of “asymptotic” logarithms.

5.8. Corollary. Let $f, g \in K^* \setminus O^*$. If $[vf] < [vg]$, then $\log |f| \sim h$ for some $h \in K^* \setminus O^*$. Conversely, $\log |f| \sim g$ implies $[vf] < [vg]$.
Proof. If \(|vf| < |vg|\), we have \(v(f'/f) \neq \sup \Psi_K\) by Theorem 3.4. Therefore \(f'/f \sim h'\) for some \(h \in K^*\) by the previous theorem. From \(f(\infty) \in \{0, \pm \infty\}\) we conclude \(\log |f(\infty)| = \pm \infty\), that is, \(v \log |f| < 0\), hence \(\log |f| \sim h \in K^* \setminus O^*\) by 5.4(c) (in some Hardy extension). Conversely, \(\log |f| \sim g\) implies \(vg = v \log |f| < 0\), hence \(v(g'/g) > v(g'/f)\) by 3.2(a), which means \([vf] < [vg]\) according to 3.2(f).

6 Miller’s dichotomy

For an expansion \(\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <, \ldots)\) of an ordered subfield \(\mathbb{R} \subseteq \mathbb{R}\), we denote by \(H(\mathcal{R}) \subseteq \mathbb{R}^{\mathbb{R}/\mathcal{N}}\) the ring of germs at infinity of \(\mathcal{R}\)-definable functions (with parameters from \(\mathbb{R}\)). If \(\mathcal{R}\) is \(o\)-minimal, then every \(f \in H(\mathcal{R})\) is eventually continuous and thus can be thought of as an element of \(C_0\); in fact, in this case, \(H(\mathcal{R})\) is a Hardy field.

6.1. Example. Let \(\mathcal{R} \subseteq \mathbb{R}\) be a subfield such that \(\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <)\) is \(o\)-minimal, that is, \(\mathbb{R}\) is an archimedean real closed field. Then \(H(\mathcal{R}) = \overline{\mathcal{R}(x)}\), the real closure of \(\mathcal{R}(x)\). In particular, \(H(\mathcal{R}) \cap \mathbb{R} = \mathbb{R}\).

Proof. Clearly, \(K = \mathbb{R}(x) \subseteq H(\mathcal{R})\). Now let \(f \in \overline{K}\). Then \(F(f) = 0\) for some non-zero \(F \in K[Y]\), say \(F = \sum_i a_i Y^i\) with \(a_i \in K\). Let \(f_1, \ldots, f_n \in K\) all “real” roots of \(F\) with \(n \in \mathbb{N}\) and \(f_1 < \cdots < f_n\). Then \(f = f_j\) for some \(j\), and the graph of \(f\) eventually looks like \(\{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} \mid \exists \eta_1, \ldots, \eta_n \in \mathbb{R} : \eta_1 < \cdots < \eta_n, \eta = \eta_j, \sum_i a_i(\xi) \eta_i^j = \cdots = \sum_i a_i(\xi) \eta_n^j = 0\}\), which is an \(\mathcal{R}\)-definable set. Hence \(f \in H(\mathcal{R})\).

Conversely, let \(f \in C_0\) be \(\mathcal{R}\)-definable. Then, by quantifier elimination, the graph of \(f\) is contained in an algebraic set \(\{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} \mid F(\xi, \eta) = 0\}\), for some non-zero bivariate polynomial \(F \in \mathbb{R}[X, Y]\). But this means \(F(x, f) = 0\), so \(f \in \overline{K}\).

The following results are due to Miller [Mi]. We say that \(\mathcal{R}\) is polynomially bounded if for every \(f \in H(\mathcal{R})\) there is \(n \in \mathbb{N}\) with \(f < x^n\) (as germs).

6.2. Proposition. Let \(\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, \ldots)\) be \(o\)-minimal and polynomially bounded, \(K = H(\mathcal{R})\) and \(f \in K^*\). Then \(f \sim cx^r\) \(\in K\) for some \(c \in \mathbb{R}^*, r \in \mathbb{R}\).

Proof. We may assume \(f(\infty) = \infty\). Then \([vf] \geq [vx]\) since \(K\) is polynomially bounded. But also \(f_{-1} \in K\) and \([vf_{-1}] \geq [vx]\), that is, \([vx] \geq [vf]\). Hence
$$[vx] = [vf]$$. We conclude that \(\text{rank} \ K = 1\) and \(vf \in vK \subseteq \mathbb{R}vx\). Therefore \(f \sim cx^r\) with \(c \in \mathbb{R}^*\) and \(r \in \mathbb{R}\). Now \(F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto f(tx)/f(x)\) is definable, hence also \(t \mapsto F(t, \infty)\). But

\[
F(t, \infty) = \lim_{x \to \infty} \frac{f(tx)}{c(tx)^r} \frac{cx^r}{f(x)} = t^r.
\]

Hence \(x^r \in K\). □

6.3. Lemma. Let \(f \in C\) be \(D\)-consistent with \(f(\infty) = 0\) and \(0 \leq s, t \in \mathbb{R}\). Then \(\lim_{x \to \infty} x(f(x + t) - f(x - s)) = 0\).

Proof. From \(\nu(\log) < 0 \leq \nu f\) we conclude \(\nu(f') > 0\) by Lemma 3.2(a). Using the Mean Value Theorem, we can write \(f(x + t) - f(x - s) = (t + s)f'(\xi(x))\) with \(x - s < \xi(x) < x + t\). Since \(f'(\infty) = f''(\infty) = 0\), we have \(\lim_{x \to \infty} (t + s)x f'(\xi(x)) = (t + s) \lim_{x \to \infty} x f'(x) = 0\), and the claim follows. □

6.4. Theorem. Let \(\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, \ldots)\) be \(o\)-minimal and not polynomially bounded. Then \(\exp: \mathbb{R} \to \mathbb{R}\) is \(\mathcal{R}\)-definable.

Proof. Set \(K = H(\mathcal{R})\). By assumption, there is \(f \in K\) with \(f(\infty) = \infty\) and \([vf] < [vx]\), hence \(\log f \sim g \in K^*\) by Corollary 5.8. Since also \(f_{-1} \in K\), we conclude \(\log f \sim g \circ f_{-1} =: l \in K\), that is, \(l = (1 + q) \log\) for some \(q \in L := K(\log)\) with \(q(\infty) = 0\). Because \(\log' \in L\), by 2.1, \(L \circ \exp\) is again a Hardy field, so \(r = q \circ \exp \in C\) is \(D\)-consistent, and \(r(\infty) = 0\). Now \(L(t, x) = l(tx) - l(x) = \log t + q(tx) \log(tx) - q(x) \log x\) is \(\mathcal{R}\)-definable for large enough \(t, x \in \mathbb{R}\), hence also the function \(t \mapsto L(t, \infty)\). But

\[
L(t, \infty) = \log t + \lim_{x \to \infty} (r(\log x + \log t) - r(\log x)) \log x = \log t
\]

by the previous lemma. Thus \(\exp = \log_{-1} \in K\). Using the functional equation \(\exp(t - s) = \exp(t)/\exp(s)\) renders \(\exp\) as an \(\mathcal{R}\)-definable function on all of \(\mathbb{R}\). □
References


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