

Pseudo Completions and Completion in Stages of o-minimal Structures.

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ABSTRACT. For an o-minimal expansion R of a real closed field and a set \mathcal{V} of $Th(R)$ -convex valuation rings, we construct a “pseudo completion” with respect to \mathcal{V} . This is an elementary extension S of R generated by all completions of all the residue fields of the $V \in \mathcal{V}$, when these completions are embedded into a big elementary extension of R . It is shown that S does not depend on the various embeddings up to an R -isomorphism.

For polynomially bounded R we can iterate the construction of the pseudo completion in order to get a “completion in stages” S of R with respect to \mathcal{V} . S is the “smallest” extension of R such that all residue fields of the unique extensions of all $V \in \mathcal{V}$ to S are complete.

Let R be a real closed field. There is a largest ordered field \hat{R} such that R is dense in \hat{R} . \hat{R} is again real closed and \hat{R} is called the completion of R (c.f. [PC]). If v is a proper real valuation on R , then \hat{R} is also the underlying field of the completion of the valued field (R, v) and \hat{R} is obtained by adjoining limits of Cauchy sequences with respect to v as explained in [Ri].

We generalize this construction as follows. Let \mathcal{V} be a set of convex valuation rings, possibly containing R itself. We construct a “smallest” real closed field containing R which has a limit for all sequences of R that become Cauchy sequences after passing to the residue field of some $V \in \mathcal{V}$. This can also be done for o-minimal expansions of real closed fields and $Th(R)$ -convex valuation rings (see section 3 for the definition of the completion in this case).

Our first result (4.1) basically says that we can adjoin the missing limits to R in any order and that the resulting elementary extension R' of R does not depend on the choices, up to an R -isomorphism. We call R' the pseudo completion of R with respect to \mathcal{V} . If R is a pure real closed field (more generally, a polynomially bounded o-minimal expansion of a real closed field), then we can compute the value groups and the residue fields of convex valuation rings of R' . Moreover for every valuation ring $V \in \mathcal{V}$ the convex hull V' of V in R' is the unique convex valuation ring of R' , lying over V .

It turns out that R' is not “complete in stages” with respect to $\mathcal{V}' := \{V' \mid V \in \mathcal{V}\}$ in general, i.e. not all residue fields of the V' are complete in general (c.f. (5.7)). Therefore, in order to get a “smallest” extension of R , which is complete in stages, we have to iterate the construction of the pseudo completion. The iteration stops at an ordinal and the resulting extension S of R is called the completion in stages of R with respect to \mathcal{V} . In (5.10), we compute the value groups and the residue fields of convex valuation rings of S . Moreover in (5.10) it is shown that every element $s \in S \setminus R$ is of the form $ax + b$ where $a, b \in R$ and $x \in S$ such that for a unique convex valuation ring W of S with $W \cap R \in \mathcal{V}$, s/\mathfrak{m}_W is the limit of a Cauchy sequence of V/\mathfrak{m}_V without limits in V/\mathfrak{m}_V ; here $\mathfrak{m}_V, \mathfrak{m}_W$ denote the maximal ideal of V, W respectively.

The explanation of the valuation theoretic notions and facts used for o-minimal expansions of fields can be found in [vdD-Lew]. Readers who are mainly interested in the case of real

closed fields may replace “o-minimal structure” by “real closed field”, “definable” by “semi algebraic” and “definable closure” by “real closure”. Moreover if $R \subseteq S$ are real closed fields and $B \subseteq S$, then the type $tp(B/R)$ of B over R can be identified with the ordering of $R[t_b \mid b \in B]$ (where the t_b are indeterminates) induced by the evaluation map $t_b \mapsto b$.

Finally we want to point out a combinatorial tool which we use in our arguments. This is a dimension in o-minimal structures, we call it the realization rank, which is coarser than the ordinary dimension associated to o-minimal structures. For real closed fields $R \subseteq S$, with $\text{tr.deg. } S/R$ finite, the realization rank of S over R is the maximal number of elements $s_1, \dots, s_k \in S$ such that $tp(s_1, \dots, s_k/R)$ is uniquely determined by the open boxes contained in it (c.f. (1.12)). We first analyze this new dimension.

1. THE REALIZATION RANK

We start with a reminder on dependence relations as in in van der Waerden’s ”Algebra” ([vdW]).

(1.1) DEFINITION. A relation $x \ll A$ between elements x and subsets A of a given set X is called a **dependence relation** if the following conditions are fulfilled:

- (D1) $x \ll \{x\}$.
- (D2) if $x \ll A$ and $A \subseteq B$ then $x \ll B$.
- (D3) if $x \ll A$ then there is a finite subset B of A , such that $x \ll B$.
- (D4) (exchange lemma)
if A is finite, $x \ll A \cup \{y\}$ and $x \not\ll A$, then $y \ll A \cup \{x\}$.
- (D5) (transitivity)
if A is finite, $x \ll A$ and $a \ll B$ for every $a \in A$, then $x \ll B$.

We rephrase this notion in terms of independent sets:

(1.2) DEFINITION. Let X be a set and let \mathcal{I} be a nonempty set of finite subsets of X . \mathcal{I} is called a **system of independence** if the following two properties hold.

- (I1) If $A \subseteq B \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
- (I2) If $A, B \in \mathcal{I}$, $x \in X \setminus B$ and if $B \cup \{x\} \in \mathcal{I}$, then $A \cup \{x\} \in \mathcal{I}$ or there is some $a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{I}$.

Observe that $\emptyset \in \mathcal{I}$ if \mathcal{I} is an independence system. Dependence relations and systems of independence describe the same concept:

(1.3) PROPOSITION. *If \mathcal{I} is a system of independence of a set X then we define a relation between elements and subsets of X by*

$$x \ll_{\mathcal{I}} A : \Leftrightarrow x \in A \text{ or there is some } A_0 \subseteq A, A_0 \in \mathcal{I} \text{ such that } A_0 \cup \{x\} \notin \mathcal{I}.$$

If \ll is a dependence relation of X then we define

$$\mathcal{I}(\ll) := \{A \mid A \text{ is finite and } a \not\ll A \setminus \{a\} \text{ for all } a \in A\}.$$

(i) *If \ll is a dependence relation of X , then $\mathcal{I}(\ll)$ is a system of independence and*

$$\ll_{\mathcal{I}(\ll)} = \ll.$$

(ii) If \mathcal{I} is a system of independence of X , then $\ll_{\mathcal{I}}$ is a dependence relation and

$$\mathcal{I}(\ll_{\mathcal{I}}) = \mathcal{I}.$$

PROOF. (i). Clearly $A \subseteq B \in \mathcal{I}(\ll)$ implies $A \in \mathcal{I}(\ll)$, hence (I1) holds for $\mathcal{I}(\ll)$. In particular $\emptyset \in \mathcal{I}(\ll)$. In order to see (I2), let $A, B \in \mathcal{I}(\ll)$ and $x \in X \setminus B$. Suppose $A \cup \{x\} \notin \mathcal{I}(\ll)$ and for all $a \in A \setminus B$, $B \cup \{a\} \notin \mathcal{I}(\ll)$. We claim that $a \ll B$ for all $a \in A$. If $a \in B$ this is true. If $a \notin B$, then $B \cup \{a\} \notin \mathcal{I}(\ll) \ni B$ and the exchange lemma imply $a \ll B$.

Hence $a \ll B$ for all $a \in A$. Again, the exchange lemma applied to $A \cup \{x\} \notin \mathcal{I}(\ll) \ni A$ gives $x \ll A$. By transitivity we get $x \ll B$. Since $x \notin B$ we get $B \cup \{x\} \notin \mathcal{I}(\ll)$ as desired.

So we know that $\mathcal{I}(\ll)$ is a system of independence and it remains to show for all $x \in X$ and all $A \subseteq X$:

$x \ll A \Leftrightarrow x \in A$ or there is some $A_0 \subseteq A$, $A_0 \in \mathcal{I}(\ll)$ such that $A_0 \cup \{x\} \notin \mathcal{I}(\ll)$.
 \Leftarrow . If $x \notin A$, $A_0 \in \mathcal{I}(\ll)$ and $A_0 \cup \{x\} \notin \mathcal{I}(\ll)$ then the exchange lemma implies $x \ll A_0$, thus $x \ll A$.
 \Rightarrow . If $x \ll A$ and $x \notin A$, then there is a finite subset A_1 of A such that $x \ll A_1$ and $x \notin A_1$. Every maximal subset A_0 of A_1 with $A_0 \in \mathcal{I}(\ll)$ has the required property.

(ii). First we check conditions (D1)-(D5) for $\ll_{\mathcal{I}}$. Clearly (D1), (D2) and (D3) hold.

(D4). Let A be finite, $x, y \in X$ with $x \ll_{\mathcal{I}} A \cup \{y\}$ and $x \not\ll_{\mathcal{I}} A$. We have to show $y \ll_{\mathcal{I}} A \cup \{x\}$. If $y = x$ this is true, so we may assume that $y \neq x$. As $x \not\ll_{\mathcal{I}} A$ we have $x \notin A \cup \{y\}$. Since $x \ll_{\mathcal{I}} A \cup \{y\}$, there is some $A_0 \subseteq A$ with $A_0 \cup \{y\} \in \mathcal{I}$ such that $A_0 \cup \{y, x\} \notin \mathcal{I}$. As $x \not\ll_{\mathcal{I}} A$ and $A_0 \in \mathcal{I}$ we have $A_0 \cup \{x\} \in \mathcal{I}$. By definition $y \ll_{\mathcal{I}} A \cup \{x\}$.

(D5). Let A, B be finite, $x \ll_{\mathcal{I}} A$ and $a \ll_{\mathcal{I}} B$ for all $a \in A$. We have to show $x \ll_{\mathcal{I}} B$. We may assume that $x \notin A \cup B$. Let $B_0 \subseteq B$ be maximal with $B_0 \in \mathcal{I}$ and let $a \in A \setminus B_0$. Suppose $B_0 \cup \{a\} \in \mathcal{I}$ for some $a \in A$. From the maximality of B_0 we get $a \notin B$. Since $a \ll_{\mathcal{I}} B$, there is some $B' \subseteq B$ with $B' \in \mathcal{I}$ and $B' \cup \{a\} \notin \mathcal{I}$. But now (I2) gives some $b' \in B' \setminus B_0$ with $B_0 \cup \{b'\} \in \mathcal{I}$, a contradiction to the maximality of B_0 . This shows that $B_0 \cup \{a\} \notin \mathcal{I}$ for all $a \in A \setminus B_0$.

As $x \ll_{\mathcal{I}} A$ there is some $A_0 \subseteq A$, $A_0 \in \mathcal{I}$ with $A_0 \cup \{x\} \notin \mathcal{I}$. Now we apply (I2) again and get $B_0 \cup \{x\} \notin \mathcal{I}$, thus $x \ll B$.

It remains to show that $\mathcal{I} = \mathcal{I}(\ll_{\mathcal{I}})$.

\subseteq . If $A \in \mathcal{I}$ and $a \in A$ then $a \not\ll_{\mathcal{I}} A \setminus \{a\}$, since for all $A_0 \subseteq A \setminus \{a\}$ we have $A_0 \cup \{a\} \in \mathcal{I}$ by (I1). Thus $A \in \mathcal{I}(\ll_{\mathcal{I}})$.

\supseteq . Let $A \in \mathcal{I}(\ll_{\mathcal{I}})$ and let $A_0 \subseteq A$ be maximal with $A_0 \in \mathcal{I}$. Suppose there is some $a \in A \setminus A_0$. Since $A \in \mathcal{I}(\ll_{\mathcal{I}})$ we have $a \not\ll_{\mathcal{I}} A \setminus \{a\}$, hence also $a \not\ll_{\mathcal{I}} A_0$. As $A_0 \in \mathcal{I}$, this means $A_0 \cup \{a\} \in \mathcal{I}$, which contradicts our choice of A_0 . \square

If \mathcal{I} is a system of independence of X with corresponding dependence relation \ll , then we write $\mathcal{I} - \text{rk}(A)$ or $\ll - \text{rk}(A)$ respectively, for the cardinality of a basis of a set $A \subseteq X$ with respect to \ll .

The Realization Rank

We always work with small subsets of a large o-minimal structure \mathfrak{M} expanding a dense linear order without endpoints. \mathfrak{M} is not mentioned always. Moreover we fix a (small) subset A of \mathfrak{M} . A is always assumed to be definably closed. For a set X , $\text{cl}(X)$ denotes the definable closure of X (in \mathfrak{M}). If $D \subseteq \mathfrak{M}$ is definably closed, then $D\langle X \rangle$ also denotes $\text{cl}(D \cup X)$.

(1.4) LEMMA. *If p is a 1-type over A and $A \subseteq B \subseteq \mathfrak{M}$, then the following conditions are equivalent.*

- (i) p has a unique extension on B .
- (ii) If p is realized in $\text{cl}(B)$ then p is realized in A .

PROOF. The set A is definably closed. Therefore each formula with parameters in A with one free variable is equivalent to a quantifier free formula of the language $\{<\}$ with parameters in A . Now the lemma follows easily. \square

(1.5) DEFINITION. If B is a subset of \mathfrak{M} and if c is an element from \mathfrak{M} , we say that c is **dominated** by B over A (or A -dominated by B) and write $c \triangleleft_A B$, if $tp(c/A)$ is realized in $\text{cl } A \cup B$; otherwise c is called **A -indominated** by B .

(1.6) COUNTEREXAMPLE. *A -dominance is not a dependence relation, since transitivity is violated. To see an example let \mathfrak{M} be a big real closed field containing \mathbb{R} , take A to be the real closure of \mathbb{Q} and let $\mu \in \mathfrak{M}$ be positive and infinitesimal over \mathbb{R} . Then*

- (a) $\mu \in A(\pi, \pi + \mu)$, thus μ is A -dominated by $\{\pi, \pi + \mu\}$
- (b) $\pi + \mu$ is A -dominated by $\{\pi\}$.
- (c) μ is A -indominated by $\{\pi\}$.

In spite of this example, the A -dominance relation leads to a dependence relation. Before introducing this relation we prove that \triangleleft_A satisfies axioms (D1)-(D4) of a dependence relation. We suppress the index A and write dominated or indominated only. The set A is always fixed and, as mentioned in the beginning, definably closed.

Certainly we have for all $c \in \mathfrak{M}$ and all $B, C \subseteq \mathfrak{M}$:

- (D1) c is dominated by $\{c\}$.
- (D2) c dominated by B , $B \subseteq C \Rightarrow c$ dominated by C .
- (D3) if c is dominated by B , then there is a finite subset B_0 of B , such that c is dominated by B_0 .

From (1.4) we know for any element $c \notin A$ the equivalence of

- (i) c is indominated by B .
- (ii) $tp(c/A \cup B)$ is the unique extension of $tp(c/A)$ on $A \cup B$.
- (iii) If $c' \in \mathfrak{M}$ such that $tp(c/A) = tp(c'/A)$, then $c' \notin \text{cl } A \cup B$.

(1.7) EXCHANGE LEMMA FOR A -DOMINANCE.

If c is indominated by B and dominated by Bd , then d is dominated by Bc .

PROOF. We search a realization of $tp(d/A)$ in $\text{cl } ABc$. Since c is dominated by Bd there is some realization $c' \in \text{cl } ABd$ of $tp(c/A)$. Because c is indominated by B we have $c' \notin \text{cl } AB$. From the exchange lemma for the algebraic relation in \mathfrak{M} we get $d \in \text{cl } ABc'$. Since c is indominated by B it follows $tp(c/AB) = tp(c'/AB)$. Let σ be an $(A \cup B)$ -automorphism of \mathfrak{M} such that $\sigma(c') = c$. Then $\sigma(d) \in \text{cl } ABc$ is a realization of $tp(d/A)$ as desired. \square

The next proposition implies a variant of transitivity for A -dominance, which we will use to define a system of independence.

(1.8) PROPOSITION. *Let I be an index set, let $\{b_i \mid i \in I\}$, C and D be sets such that for each $i \in I$, b_i is indominated by $C \cup \{b_j \mid j \neq i\}$. Suppose b_i is dominated by $C \cup D$ for every $i \in I$. Then $tp((b_i)_{i \in I}/A \cup C)$ is realized in $\text{cl } A \cup C \cup D$. More precisely: If b'_i is a realization of $tp(b_i/A)$ in $\text{cl } A \cup C \cup D$, then $(b'_i)_{i \in I}$ is a realization of $tp((b_i)_{i \in I}/A \cup C)$.*

PROOF. We have $b_i \neq b_j$ if $i \neq j$ and it is enough to prove the Proposition for finite I . We do an induction on $n = \text{card } I$:

$n = 1$: suppose b is dominated by $C \cup D$, indominated by C and b' realizes $tp(b/A)$ in $\text{cl } A \cup C \cup D$. Since b is indominated by C the type $tp(b/A \cup C)$ is realized by b' too.

$n \rightarrow n + 1$. Suppose $\{b_1, \dots, b_{n+1}\}$ is indominated by C and b_i is dominated by $C \cup D$. Let $b'_1, \dots, b'_{n+1} \in \text{cl } A \cup C \cup D$ be realizations of $tp(b_1/A), \dots, tp(b_{n+1}/A)$ respectively. By the induction hypothesis we have $tp(b_1, \dots, b_n/A \cup C) = tp(b'_1, \dots, b'_n/A \cup C)$.

Let σ be an $A \cup C$ -automorphism of \mathfrak{M} such that $\sigma(b_i) = b'_i$ ($1 \leq i \leq n$). Since b_{n+1} is indominated by $C \cup \{b_1, \dots, b_n\}$, we see that $\sigma(b_{n+1})$ is indominated by $C \cup \{b'_1, \dots, b'_n\}$, that is $tp(\sigma(b_{n+1})/A \cup C \cup \{b'_1, \dots, b'_n\}) = tp(b'_{n+1}/A \cup C \cup \{b'_1, \dots, b'_n\})$. Hence (b'_1, \dots, b'_{n+1}) is a realization of $tp(b_1, \dots, b_{n+1}/A \cup C)$. \square

(1.9) COROLLARY AND DEFINITION. *Let $A, C \subseteq \mathfrak{M}$ and let A be definably closed. For elements $x \in \mathfrak{M}$ and subsets B of \mathfrak{M} we define $x \triangleleft_{A,C} B : \Leftrightarrow x \triangleleft_A (C \cup B)$. Then $\triangleleft_{A,C}$ satisfies properties (D1)-(D4) of a dependence relation (cf (1.1)). Moreover*

$$\mathcal{I}(A, C) := \{B \subseteq \mathfrak{M} \mid B \text{ is finite and for all } b \in B \text{ we have } b \not\triangleleft_{A,C} B \setminus \{b\}\}.$$

is a system of independence. The dependence relation corresponding to $\mathcal{I}(A, C)$ as explained in (1.3) is denoted by $\ll_{A,C}$. The dimension associated with $\ll_{A,C}$ is denoted by $\text{rk}_{A,C}$ and is called the Realization rank with respect to A, C .

PROOF. By (1.7), properties (D1)-(D4) of a dependence relation hold for $\triangleleft_{A,C}$. In particular property (I1) of an independence system holds for $\mathcal{I}(A, C)$ and it remains to show that also property (I2) of an independence system holds for $\mathcal{I}(A, C)$.

To see this let $B, D \in \mathcal{I}(A, C)$ and let $x \notin D$. Suppose $B \cup \{x\} \notin \mathcal{I}(A, C)$ and $D \cup \{b\} \notin \mathcal{I}(A, C)$ for all $b \in B$. Since $\triangleleft_{A,C}$ satisfies (D1)-(D4) this means $x \triangleleft_{A,C} B$ and $b \triangleleft_{A,C} D$ for all $b \in B$. As $B \in \mathcal{I}(A, C)$ we can apply (1.8):

Let $B = \{b_1, \dots, b_n\}$ and let F be an $A \cup C$ -definable map, such that $F(b_1, \dots, b_n)$ is a realization of $tp(x/A)$. From (1.8) we know that $tp(b_1, \dots, b_n/A \cup C)$ is realized in $\text{cl } A \cup C \cup D$ by some n -tuple (b'_1, \dots, b'_n) . If σ is an $A \cup C$ -automorphism of \mathfrak{M} such that $\sigma(b_i) = b'_i$ then $\sigma(F(b_1, \dots, b_n)) = F(b'_1, \dots, b'_n)$ is a realization of $tp(x/A)$ in $\text{cl } A \cup C \cup D$.

Hence $x \triangleleft_{A,C} D$ and $D \cup \{x\} \notin \mathcal{I}(A, C)$. \square

If the set C is contained in A we write \ll_A and rk_A instead of $\ll_{A,C}$ and $\text{rk}_{A,C}$. A set B is called $\ll_{A,C}$ -independent if every finite subset of B is in $\mathcal{I}(A, C)$.

(1.10) PROPOSITION. *We have for every set $B \subseteq \mathfrak{M}$:*

(i) *B is $\ll_{A,C}$ -independent if and only if $b \not\triangleleft_{A,C} B \setminus \{b\}$ for all $b \in B$.*

(ii) *For all $x \in \mathfrak{M}$, $x \ll_{A,C} B \Leftrightarrow x \triangleleft_{A,C} B_0$ for an $\ll_{A,C}$ -independent subset B_0 of B .*

(iii)

$$\text{rk}_{A,C}(B) = \min\{\text{card } B_0 \mid B_0 \subseteq B \text{ and } b \triangleleft_{A,C} B_0 \text{ for all } b \in B\}$$

PROOF. (i) holds by definition of $\ll_{A,C}$ and since $\triangleleft_{A,C}$ satisfies (D1)-(D4) and (ii) is implied by (i).

(iii). \geq holds, since by (ii), for a $\ll_{A,C}$ -basis B_0 of B we have $b \triangleleft_{A,C} B_0$ for all $b \in B$.

Conversely let B' be a $\ll_{A,C}$ -basis of B and let $B_0 \subseteq B$, such that each $b \in B$ is A -dominated by $B_0 \cup C$. By (1.8) the type of B' over $A \cup C$ is realized in $\text{cl}(A \cup C \cup B_0)$. Since $\dim B'/A \cup C = \text{card } B'$ it follows that $\dim B_0/A \cup C \geq \text{card } B'$. Hence $\text{rk}_{A,C} B = \text{card } B' \leq \dim B_0/A \cup C \leq \text{card } B_0$. \square

A set $B_0 \subseteq B$, which is minimal with the property

$$b \in B \Rightarrow b \text{ is dominated by } C \cup B_0$$

need not be indominated over C . Look at the following example.

Examples

Let R_0 be the real closure of \mathbf{Q} in \mathbb{R} and let μ be some positive infinitesimal. Then we have $\text{rk}_{R_0}(R_0(\pi + \mu, \pi)/R_0) = 2$. But the set $\{\pi + \mu, \pi\}$ is not an R_0 -dominance basis of $R_0(\pi + \mu, \pi)$ over R_0 .

In particular \ll_{R_0} is different from \triangleleft_{R_0} .

Furthermore we have

$$\text{rk}_{R_0, \mu}(\{\pi + \mu, \pi\}) = 1 = \text{rk}_{R_0}(\{\pi + \mu, \pi\}/R_0)$$

and

$$\text{rk}_{R_0, \{\pi, \pi + \mu\}}(\mu) = 0 \neq 1 = \text{rk}_{R_0}(\mu)$$

That is: the symmetry

$$\text{rk}_{A,D}(B) = \text{rk}_A B \Rightarrow \text{rk}_{A,B}(D) = \text{rk}_A(D)$$

does not hold in general.

If A is a subset of \mathbb{R} , and B is an arbitrary set, then $\text{rk}_{A, \mathbb{R}}(B) \leq 1$.

Hence, if $p = \text{tp}(d, \pi/R_0)$ and (α, β) is another realization of p , we have

$$\text{rk}_{A, \mathbb{R}}(\{\alpha, \beta\}) < \text{rk}_{R_0}(d, \pi)$$

The next proposition gives a geometric interpretation of $\text{rk}_{A,C}$.

(1.11) DEFINITION. Let R be ω -minimal and let C be subset of an elementary extension of R . Let p be an n -type over R . We say that p is a **box type over C** if the (R -definable) open boxes, which contain p build a basis of neighborhoods for an n -type of $S_n(R \cup C)$.

If $C \subseteq R$, we just say p is a box type.

So if p is a box type over C , then p has a unique extension on $S_n(R \cup C)$ and the open R -definable boxes containing p imply this extension.

Note that if $\bar{a} \in R^n$, then $\{\text{tp}(\bar{a}/R)\}$ is a neighborhood of $\text{tp}(\bar{a}/R)$, which does not contain an open box.

(1.12) PROPOSITION. *If $p \in S_n(R)$ then the following conditions are equivalent:*

(i) *For some (hence for each) realization $\bar{\alpha}$ of p we have $\text{rk}_{R,C}(\bar{\alpha}) = n$.*

(ii) *p is a box type over C .*

(iii) *$\alpha_1, \dots, \alpha_n \notin R$ and $\text{tp}(\bar{\alpha}/R \cup C)$ is the unique n -type over $R \cup C$ containing each $\text{tp}(\alpha_i/R)$.*

PROOF. Obviously each of the conditions (i) and (ii) imply $\dim p = n$.

(i) \Rightarrow (ii). By induction on n . If $n = 1$, then p is omitted in $R\langle C \rangle$, thus (ii) holds. For the induction step, let $\bar{\alpha}$ to be an $n - 1$ -tuple and let β be an element, such that p is realized by $\bar{\alpha} \hat{\ } \beta$ with $\text{rk}_{R,C}(\bar{\alpha} \hat{\ } \beta) = n$. By the induction hypothesis, $\text{tp}(\bar{\alpha}/R)$ is a box type over C .

Let X be an $R \cup C$ -definable set which contains $tp(\bar{\alpha}, \beta/R \cup C)$. Since $\dim \bar{\alpha}, \beta/R \cup C = n$, we can suppose that X is an open cell $(F, G)_Y$, where F, G and Y are $R \cup C$ -definable. We have $F(\bar{\alpha}) < \beta$. As $tp(\beta/R)$ is omitted in $R\langle C\bar{\alpha} \rangle$, there is some $a_1 \in R$ with $F(\bar{\alpha}) \leq a_1 < \beta$. Similar we can find some $a_2 \in R$ with $\beta < a_2 \leq G(\bar{\alpha})$. Since $tp(\bar{\alpha}/R)$ is a box type over C , there is an open R -definable box $Y_0 \subseteq \{\bar{b} \in Y \mid F(\bar{b}) \leq a_1, a_2 \leq G(\bar{b})\}$ with $tp(\bar{\alpha}/R) \in Y_0$. Finally $Y \times (a_1, a_2)$ is an open box, which contains $tp(\bar{\alpha}, \beta/R \cup C)$ and which is contained in X .

(ii) \Rightarrow (i). We do again an induction on n . If $n = 1$, then (ii) implies that p is omitted in $R\langle C \rangle$, thus $\text{rk}_{R,C}(\alpha) = 1$ for all realizations α of p . Assume $p \in S_n(R)$ is a box type over C and $\bar{\alpha}\bar{\beta}$ is a realization of p . Certainly $tp(\bar{\alpha}/R)$ is a box type over C and by the induction hypothesis $\text{rk}_{R,C}(\bar{\alpha}) = n - 1$. We have to show that $tp(\beta/R)$ is omitted in $R\langle C\bar{\alpha} \rangle$: Let F be an $R \cup C$ -definable map, say $F(\bar{\alpha}) < \beta$. Let $Y \subseteq R^{n-1}$ be an open box and let $a_1 < a_2 \in R$ with $p \in Y \times (a_1, a_2)$ and $Y \times (a_1, a_2) \subseteq \{\bar{b}, b' \in R^n \mid F(\bar{b}) < b'\}$. That is $F(\bar{b}) \leq a_1$ for all $\bar{b} \in Y$, hence $F(\bar{\alpha}) \leq a_1 < \beta$.

(ii) \Leftrightarrow (iii). If p_1, \dots, p_n are the projections of p and each p_i is a cut over R , then the intersection of all open boxes containing p in $S_n(R \cup C)$ is the set of all n -types $q \in S_n(R \cup C)$ which contain p_1, \dots, p_n . \square

(1.13) COROLLARY. *If R is a real closed field and $p \in \text{Sper } R[t]$, $t = (t_1, \dots, t_n)$ is a box type such that R is archimedean in the real closure of p then p is minimal and maximal in $\text{Sper } R[t]$.*

PROOF. Since $\dim p = n$, p is minimal in $\text{Sper } R[t]$. On the other hand, if $q \in \text{Sper } R[t]$ is different from p then there is an open box B containing p and not containing q . Since R is archimedean in the real closure of p , we can find a smaller open box B' containing p with $\overline{B'} \subseteq B$. Hence p does not specialize to q . \square

Observe that the converse of (1.13) fails in general. The reason is that a semi algebraic homeomorphism $R^n \rightarrow R^n$ respects the topology of $\text{Sper } R[t]$ - hence minimal, maximal points are mapped to minimal, maximal points - but not the property “ p is a box type”.

In other words, box types can not be detected with the topology of $\text{Sper } R[t]$.

(1.14) PROPOSITION. *Let $R \prec \mathfrak{M}$ and let $B, C \subseteq \mathfrak{M}$ such that B is $\ll_{R,C}$ -independent. Then $tp(B/R \cup C)$ is the unique extension of $tp(B/R)$. In particular $tp(B/R \cup C)$ is an heir of $tp(B/R)$ (c.f. [Ho], p. 292, for the definition of “heir”)*

PROOF. We may assume that $B = \{b_1, \dots, b_n\}$ is finite and we do an induction on n . If $n = 1$, then $tp(b_1/R \cup C)$ is the unique extension of $tp(b_1/R)$, since $b_1 \not\prec_R C$. In the induction step we have: $tp(b_1, \dots, b_n/R \cup C)$ is the unique extension of $tp(b_1, \dots, b_n/R)$ (from the induction hypothesis) and $tp(b_{n+1}/R \cup C \cup \{b_1, \dots, b_n\})$ is the unique extension of $tp(b_{n+1}/R)$ (since B is $\ll_{R,C}$ -independent). These two properties are equivalent to the property that $tp(b_1, \dots, b_{n+1}/R \cup C)$ is the unique extension of $tp(b_1, \dots, b_{n+1}/R)$. \square

Behavior of \ll_R under base change.

First a reminder on the functional version of the Marker-Steinhorn Theorem.

(1.15) THEOREM. *Let $R \prec S$ be a tame extension of o-minimal expansions of fields (“tame” means that every $s \in S$, which is R -bounded is infinitely close to an element of R with respect to R). Let V be the convex hull of R in S and let $\lambda : S \rightarrow R \cup \{\infty\}$ be the place according*

to V . Furthermore let $X \subseteq S^n$ and $F : X \rightarrow S$ be definable in (S, V) with parameters from S . For a subset Y of S^n let $H(Y) := \bigcup_{y \in Y} y + \mathfrak{m}_V^n$ denote the set of all points of S^n which are infinitely close to a point of Y with respect to R . Then

(i) The composed map

$$\lambda F : F^{-1}(V) \cap R^n \rightarrow S^n \xrightarrow{F} S \xrightarrow{\lambda} R \cup \{\infty\}$$

is R -definable. λF is the unique map $F^{-1}(V) \cap R^n \rightarrow R$ with the property $(\lambda F)(\bar{a}) = \lambda(F(\bar{a}))$ ($\bar{a} \in R \cap F^{-1}(V)$).

(ii) There is a decomposition $R^n = E \cup D \cup F \cup C$ of R^n in R -definable sets, such that:

- (a) F is positive infinite on $H(D)$.
- (b) F is negative infinite on $H(F)$.
- (c) $F - (\lambda F)_S$ is infinitesimal on $H(C)$ and
- (d) $\dim E < n$.

PROOF. This is [Ma-St], Theorem 3.3. □

(1.16) PROPOSITION. Let $R \prec S$ be o -minimal expansions of fields and let B be from an elementary extension of S such that $b \not\prec_R S$ for all $b \in B$.

- (i) If B is \ll_S -independent then B is \ll_R -independent.
- (ii) If R is tame in S , then B is \ll_S -independent if and only if B is \ll_R -independent.

PROOF. (i). Suppose B is \ll_S -independent and not \ll_R -independent. By induction on n we may assume that there are b, b_1, \dots, b_n such that $\{b_1, \dots, b_n\}$ is \ll_R -independent and such that $F(b_1, \dots, b_n)$ and b realize the same cut of R for some R -definable map $F : R^n \rightarrow R$. By assumption, $F(b_1, \dots, b_n)$ and b realize the same cut of S , hence $\{b, b_1, \dots, b_n\}$ is \ll_S -dependent, a contradiction.

(ii). Now suppose R is tame in S . Let V be the convex hull of R in S and let $\lambda : S \rightarrow R \cup \{\infty\}$ be the place according to V . Suppose B is \ll_R -independent and not \ll_S -independent. Again, by induction we find $b, b_1, \dots, b_n \in B$ such that $\{b_1, \dots, b_n\}$ is \ll_S -independent, but for some S -definable map $F : S^n \rightarrow S$, the element $F(b_1, \dots, b_n)$ induces the same cut over S as b . Let $Z := F^{-1}(V) \cap R^n$ and $\lambda F : Z \rightarrow R$ as in (1.15)(i). Let $R^n = E \cup D \cup F \cup C$ of R^n be a decomposition as in (1.15)(ii).

Since $F(b_1, \dots, b_n) = b$ is R -bounded, there is an S -definable set Z_0 such that F is R -bounded on Z and such that $Z_0 \in tp(b_1, \dots, b_n)$. By (1.12) and since $\{b_1, \dots, b_n\}$ is \ll_S -independent, there is an S -definable, open box O such that $\overline{O} \subseteq Z_0 \setminus E$ and such that $O \in tp(b_1, \dots, b_n)$. Since $b_i \not\prec_R S$ ($1 \leq i \leq n$), we may shrink O so that O is R -definable. From (1.15)(ii) we get that $F - (\lambda F)_S$ has values in \mathfrak{m}_V on $H(\overline{O}) \supseteq \overline{O}_S$. But then also $F(b_1, \dots, b_n) - \lambda F(b_1, \dots, b_n)$ is infinitesimal with respect to R . Since the cut of b over R is not definable and $F(b_1, \dots, b_n)$ realizes this cut, also $\lambda F(b_1, \dots, b_n)$ realizes this cut. Since λF is R -definable, $\{b, b_1, \dots, b_n\}$ is \ll_R -dependent. □

(1.17) LEMMA. Let $R \prec S$ be o -minimal expansions of fields and let b be from an elementary extension of S .

- (i) If S is dense in $S\langle b \rangle$ and $b \not\prec_R S$ then R is dense in $R\langle b \rangle$.
- (ii) If R is dense in $R\langle b \rangle$ and S does not contain infinitesimal elements with respect to R , then S is dense in $S\langle b \rangle$.

PROOF. (i). Suppose there are $\alpha, \beta \in R\langle b \rangle$, $\alpha < \beta$ with $(\alpha, \beta) \cap R = \emptyset$. We may assume that $\alpha, \beta \notin R$. Since S is dense in $S\langle b \rangle$ there is some $s \in S$ with $\alpha < s < \beta$, thus $tp(s/R) = tp(\alpha/R)$. Since $\alpha \in R\langle b \rangle \setminus R$ there is an R -definable map $f : R \rightarrow R$ such that $f(\alpha) = b$. Hence $f(s)$ realizes $tp(b/R)$, a contradiction.

(ii). Suppose S is not dense in $S\langle b \rangle$ and S does not contain infinitesimal elements with respect to R . Take $\alpha, \beta \in S\langle b \rangle \setminus S$ with $\alpha < \beta$ such that $(\alpha, \beta) \cap S = \emptyset$, in particular $tp(\alpha/S) = tp(\beta/S)$. Let $f : S \rightarrow S$ be S -definable such that $f(\alpha) = b$. Then f is strictly monotonic on the realizations of $tp(\alpha/S)$, hence $\gamma := f(\beta) \neq b$ is a realization of $tp(b/S)$. Say $b < \gamma$. Since S does not contain infinitesimal elements with respect to R there is some $m \in R$ such that $0 < m < \gamma - b$. Then $b < b + m < \gamma$ and there is no element in R between b and $b + m$. \square

(1.18) PROPOSITION. *Let $R \prec S$ be an o-minimal expansions of fields and let $B, D \subseteq S$. Let B be \ll_R -independent such that R is neither dense nor tame in $R\langle b \rangle$ for all $b \in B$. If D is another \ll_R -independent set such that R is dense in $R\langle d \rangle$ for each $d \in D$, then $R\langle B \rangle$ is dense in $R\langle B \cup D \rangle$ and $B \cup D$ is \ll_R -independent.*

PROOF. We may assume that $B = \{b_1, \dots, b_n\}$ and $D = \{d_1, \dots, d_k\}$ are finite. First observe that R is archimedean in $R\langle B \rangle$, otherwise by induction, b_n is infinitely close to some $c \in R\langle b_1, \dots, b_{n-1} \rangle$ and R is archimedean in $R\langle b_1, \dots, b_{n-1} \rangle$. But then either b_n has a definable type over R or c and b_n have the same type over R , a contradiction to our assumption.

By (1.17)(ii) applied to $R \prec R\langle B \rangle$ and d_1 , $R\langle B \rangle$ is dense in $R\langle B, d_1 \rangle$. By (1.17)(ii) applied to $R \prec R\langle B, d_1 \rangle$ and d_2 , $R\langle B, d_1 \rangle$ is dense in $R\langle B, d_1, d_2 \rangle$. Continuing in this way we see that $R\langle B \rangle$ is dense in $R\langle B \cup D \rangle$.

Now we prove by induction on n that $B \cup D$ is \ll_R -independent. Suppose we know that $\{b_1, \dots, b_{n-1}\} \cup D$ is \ll_R -independent. Suppose $tp(b_n/R)$ is realized in $R\langle \{b_1, \dots, b_{n-1}\} \cup D \rangle$. Since B is \ll_R -independent, $tp(b_n/R\langle b_1, \dots, b_{n-1} \rangle)$ is realized in $R\langle \{b_1, \dots, b_{n-1}\} \cup D \rangle$. Since $R\langle b_1, \dots, b_{n-1} \rangle$ is dense in $R\langle \{b_1, \dots, b_{n-1}\} \cup D \rangle$, also $R\langle b_1, \dots, b_{n-1} \rangle$ is dense in $R\langle B \rangle$. By (1.17)(i), R is dense in $R\langle b_n \rangle$ a contradiction. \square

2. V-LIMITS

If X is a totally ordered set, then a cut p of X is a tuple $p = (p^L, p^R)$ with $X = p^L \cup p^R$ and $p^L < p^R$. If $Y \subseteq X$ then Y^+ denotes the cut p of X with $p^R = \{x \in X \mid x > Y\}$. Y^+ is called the upper edge of Y . Similarly the lower edge Y^- of Y is defined.

(2.1) DEFINITION. Let p be a cut of an ordered abelian group K , The convex subgroup

$$G(p) := \{a \in K \mid a + p = p\}$$

of K is called the **invariance group** of p (here $a + p := (a + p^L, a + p^R)$).

If K is an ordered field, then the convex valuation ring

$$V(p) := \{a \in K \mid a \cdot G(p) \subseteq G(p)\}$$

is called the **invariance valuation ring** of p . If $s \notin K$ is from an ordered field extension of K then we write $G(s/K)$ and $V(s/K)$ for the invariance group and the invariance ring of the cut induced by s on K .

(2.2) DEFINITION. Let K be a divisible ordered abelian group and let p be a cut of K . We may define the **signature** of p as

$$\text{sign } p := \begin{cases} 1 & \text{if there is a convex subgroup } G \text{ of } K \text{ and some } a \in K \text{ with } p = a + G^+ \\ -1 & \text{if there is a convex subgroup } G \text{ of } K \text{ and some } a \in K \text{ with } p = a - G^+ \\ 0 & \text{otherwise} \end{cases}$$

Since K is divisible we can not have $a + G^+ = b + H^-$ for $a, b \in K$ and convex subgroups G, H of K . Hence the signature is well defined.

(2.3) DEFINITION. Let K be an ordered field and let $V \subseteq R$ be a convex valuation ring with maximal ideal \mathfrak{m}_V . A cut p of K is called a **V -limit** if $\text{sign } p = 0$ and if there is some $a \in K^*$ such that $G(p) = a \cdot \mathfrak{m}_V$. Observe that $V(p) = V$ in this case.

If in addition $G(p) = \mathfrak{m}_V$ and $\mathfrak{m}_V^+ \leq p \leq V^+$, then p is called a **proper V -limit**. Observe that $\mathfrak{m}_V^+ < p < V^+$ in this case, as $\text{sign } p = 0$.

An element b from an ordered field extension L of K is called a (proper) V -limit if $b \notin K$ and if the cut of b induced on K is a (proper) V -limit.

The next proposition states some reformulations of the notion ‘‘proper V -limit’’. First some notations. If K is an ordered field, then a sequence $(a_\alpha)_{\alpha < \lambda}$ of elements of K is called a Cauchy sequence, if it is a Cauchy sequence with respect to the order topology of K . Observe that for a non trivial convex valuation ring V of K , a Cauchy sequence with respect to V in the valuation theoretic sense (c.f. [Ri]) is a Cauchy sequence in our sense. Moreover if $(a_\alpha)_{\alpha < \lambda}$ is a Cauchy sequence, then a subsequence of $(a_\alpha)_{\alpha < \lambda}$ is a Cauchy sequence with respect to V in the valuation theoretic sense.

An element b from an ordered field extension of K is the limit of a Cauchy sequence $(a_\alpha)_{\alpha < \lambda}$ of K if

$$\forall \varepsilon \in K, \varepsilon > 0 \exists \alpha_0 < \lambda \forall \alpha > \alpha_0 |b - a_\alpha| < \varepsilon.$$

If T is an o-minimal expansion of the theory of real closed fields, then a convex valuation ring V of a model R of T is called T -convex, if T is the convex hull of an elementary restriction of R . In this case, every maximal subfield $K \subseteq V$ is an elementary restriction of R (c.f. [vdD-Lew]).

If T is the theory of real closed fields, then all convex subrings of R are T -convex.

(2.4) PROPOSITION. Let $L \subseteq M$ be an extension of ordered fields, let $W \subseteq M$ be a convex subring and let $V := W \cap L$. Let $K \subseteq W$ be a subfield such that $K/\mathfrak{m}_W = V/\mathfrak{m}_W$. The following are equivalent for every $b \in M$:

- (i) b is a proper V -limit.
- (ii) $b \in W^*$ and b/\mathfrak{m}_W is the limit of a Cauchy sequence of V/\mathfrak{m}_V without limits in V/\mathfrak{m}_V .
- (iii) b is the limit of a Cauchy sequence of K without limits in K .
- (iv) $b \notin K$ and K is dense in the ordered group $K + bK$.
- (v) $b \notin K$, $\text{sign } b/K = 0$ and $G(b/K) = \{0\}$.

Moreover b is a V -limit if and only if there are $a_1, a \in L$, $a \neq 0$ such that $a_1 + ab$ is a proper V -limit. If b is a V -limit and a V' -limit for some convex valuation ring, then $V = V'$.

If L, M are models of an o-minimal expansion T of the theory of real closed fields, $K, L \prec M$ and if V is T -convex, then K is dense in $K\langle b \rangle$ if b is a proper V -limit.

PROOF. We may assume that $b > 0$.

(i) \Rightarrow (v). First we prove that $b \notin V + \mathfrak{m}_W$. Suppose $b - a \in \mathfrak{m}_W$ for some $a \in V$, say $a < b$. Since $\text{sign } b/L = 0$ and $G(b/L) = \mathfrak{m}_V$, there is some $c \in V$, $c > \mathfrak{m}_V$ with $a + c < b$. Hence $\mathfrak{m}_V < c < b - a \in \mathfrak{m}_W$ in contradiction to $V = L \cap W$.

This proves $b \notin K$ and for all $c \in K$, $a \in V$ with $c - a \in \mathfrak{m}_W$ we have $c < b$ iff $a < b$.

Let $c \in K$, $c > 0$. We prove that $c \notin G(b/K)$. Let $v \in V$ with $c - v \in \mathfrak{m}_W$, say $v < c$. Then $v > \mathfrak{m}_V$, since $c > 0$. Since $G(b/L) = \mathfrak{m}_V$ there is some $a \in V^{>0}$ such that $a < b < a + v$. Let $c_1 \in K$ with $c_1 - a \in \mathfrak{m}_W$. Then also $a + v - (c + c_1) \in \mathfrak{m}_W$ and by what we have shown, $a < b < a + v$ implies $c_1 < b < c_1 + c$. Thus $c \notin G(b/K)$ as desired.

It remains to show that $\text{sign}(b/K) = 0$, say $\text{sign}(b/K) \geq 0$. Since $G(b/K) = 0$ it is enough to find for every element $c \in K$ with $c < b$ an element $c_1 \in K$, $c_1 > 0$ with $c + c_1 < b$. Let $a \in V$ with $c - a \in \mathfrak{m}_W$. Then $a < b$ and from $\text{sign}(b/L) = 0$, $G(b/L) = \mathfrak{m}_V$ we get some $v \in V$, $v > \mathfrak{m}_V$ with $a + v < b$. Take $c_1 \in K$ with $c_1 - v \in \mathfrak{m}_W$. Then $c_1 > 0$ and $c + c_1 < b$ since $c + c_1 - (a + v) \in \mathfrak{m}_W$.

(v) \Rightarrow (i). First we prove that $b \notin L + \mathfrak{m}_W$. Suppose $b - a \in \mathfrak{m}_W$ for some $a \in L$. Since $\text{sign}(b/K) = 0$, there is some $v \in V$ with $b < v$. But then also $a \in V$. Let $c \in K$ with $a - c \in \mathfrak{m}_W$. Then $b - c \in \mathfrak{m}_W$ and there is no element in K between b and c . This implies that the cut of b over K is definable, a contradiction to $\text{sign } b/K = 0$.

Hence $b \notin L + \mathfrak{m}_W \supseteq V + \mathfrak{m}_W = K + \mathfrak{m}_W$. We prove $G(b/L) = \mathfrak{m}_V$. First let $v \in \mathfrak{m}_V$, $v \geq 0$ and suppose there is some $l \in L$ with $b < l < b + v$. Then $l - b \in \mathfrak{m}_W$ in contradiction to $b \notin L + \mathfrak{m}_W$. Hence $\mathfrak{m}_V \subseteq G(b/L)$. Conversely let $a \in V$, $a > \mathfrak{m}_V$ and take $c \in K$ with $a - c \in \mathfrak{m}_W$. Then $c > 0$ and since $G(b/K) = 0$ there is some $c_1 \in K$, $c_1 > 0$ with $c_1 < b < c_1 + c$. Let $a_1 \in V$ with $a_1 - c_1 \in \mathfrak{m}_W$. Then $a_1 < b < a_1 + a$, hence $a \notin G(b/L)$.

Thus we know $G(b/L) = \mathfrak{m}_V$ and it remains to show that $\text{sign}(b/L) = 0$. Suppose there is some $a \in V$ such that the cut η of b over L is $a \pm \mathfrak{m}_V^\dagger$, say $\eta = a + \mathfrak{m}_V^\dagger$. Let $c \in K$ with $a - c \in \mathfrak{m}_W$. Then $c < b$. Since $\text{sign}(b/K) = 0$ there is some $c_1 \in K$, $c_1 > 0$ with $c + c_1 < b$. Let $a_1 \in V$ with $c_1 - a_1 \in \mathfrak{m}_W$. Then $a + a_1 < b$ and $a_1 > \mathfrak{m}_W \supseteq \mathfrak{m}_V$, in contradiction to $\eta = a + \mathfrak{m}_V^\dagger$.

All other implications are easy reformulations of each other, we omit the details.

Clearly b is a V -limit if and only if there are $a_1, a \in L$, $a \neq 0$ such that $a_1 + ab$ is a proper V -limit. If V' is another convex subring of L , then there is some $a \in L$ such that $a \cdot \mathfrak{m}_V = \mathfrak{m}_{V'}$. This is only possible if $V = V'$.

Now let L, M be models of an o-minimal expansion T of the theory of real closed fields, $L \prec M$ and let V be T -convex. Suppose b is a proper V -limit. Suppose K is not dense in $K\langle b \rangle$. Let $a, c \in K\langle b \rangle$ with $a < c$ and $(a, c) \cap K = \emptyset$. We may assume that $a, c \notin K$. There is a K -definable map $f : K \rightarrow K$ such that $f(a) = b$. As a and c realize the same cut over K , b and $f(c)$ realize the same cut over K . Moreover f is strictly monotonic in $[a, c] \subseteq K\langle b \rangle$, say $b < f(c)$. Since $\text{sign } b/K = 0$ there is some $d \in K$ with $0 < d < f(c) - b$. As $(b, f(c)) \cap K = \emptyset$ we get $d \in G(b/K)$, a contradiction to $G(b/K) = 0$. \square

REMARKS. Observe that an ordered field K need not be dense in $K(b)$ if b is the limit of a Cauchy sequence of K without limits in K . For example if $K = \mathbb{Q}$, $\varepsilon \neq 0$ is infinitesimal and $b = \sqrt{2} + \varepsilon$. Also, a field K as in (2.4) can not be found inside V in general. For example if $L = \mathbb{Q}(\sqrt{2} + \varepsilon)$, where ε is infinitesimal and V is the convex hull of \mathbb{Q} in L . Then \mathbb{Q} is the unique subfield of V and $V/\mathfrak{m}_V \cong \mathbb{Q}(\sqrt{2})$.

Here another reformulation of the notion ‘‘proper V -limit’’ in terms of so called distinguished Cauchy sequences as explained in [Ri], section D:

If (K, V_0) is a valued field, then a sequence $(a_\alpha)_{\alpha < \lambda}$ is called distinguished Cauchy sequence if $(a_\alpha)_{\alpha < \lambda}$ is a pseudo Cauchy sequence of the valued field (K, V_0) , such that $a_\alpha \in V_0$ for all α and such that for some valuation ring V of K with $V_0 \subsetneq V$ the $(a_\alpha - a_\beta)/V_0^*$ are unbounded in the convex subgroup V^*/V_0^* of K^*/V_0^* .

We call V the valuation ring associated to $(a_\alpha)_{\alpha < \lambda}$.

(2.5) COROLLARY. *In the situation of (2.4), let $V_0 \subsetneq V$ be another convex valuation ring. Then b is a proper V -limit if and only if b is the pseudo limit of a distinguished pseudo Cauchy sequences of the valued field (L, V_0) which does not have a limit in L and which has V as associated valuation ring.*

PROOF. Easily from (2.4). \square

(2.6) PROPOSITION. *Let T be an o-minimal expansion of fields in the language \mathcal{L} , let $R \models T$ and let \mathcal{V} be a set of T -convex valuation rings of R . For each $V \in \mathcal{V}$ let $K_V \subseteq V$ be a maximal subfield of V . Let $S \succ R$ and for each $V \in \mathcal{V}$ let $B_V \subseteq S$ be a set of proper V -limits.*

Then $\bigcup_{V \in \mathcal{V}} B_V$ is \ll_R -independent if and only if B_V is \ll_{K_V} -independent for all $V \in \mathcal{V}$.

PROOF. We write $B := \bigcup_{V \in \mathcal{V}} B_V$. By (2.4) each $b \in B_V$ has signature 0 over K_V . Hence $b \not\ll_{K_V} R$ for all $b \in B_V$. So if B is \ll_R -independent, then B_V is \ll_R -independent and by (1.16), B_V is \ll_{K_V} -independent for all $V \in \mathcal{V}$.

For the converse we may assume that \mathcal{V} is finite, say $\mathcal{V} = \{V_1, \dots, V_n\}$ and $V_1 \subsetneq \dots \subsetneq V_n$. Let $B_i \subseteq B_{V_i}$ be finite. It is enough to prove by induction on n that $B = B_1 \cup \dots \cup B_n$ is \ll_R -independent if each B_i is $\ll_{K_{V_i}}$ -independent. If $n = 1$, then we know this from (1.16)(ii).

$n \rightarrow n + 1$. Let $L_i \subseteq V_i$ be a maximal subfield of V_i with $L_1 \subseteq \dots \subseteq L_{n+1}$. From the case $n = 1$ we know that each B_i is \ll_R -independent. By what we have above, B_i is \ll_{L_i} -independent. By (2.4), each $b \in B_i$ is a $V_i \cap L_{n+1}$ -limit. By the induction hypothesis, $B_1 \cup \dots \cup B_n$ is $\ll_{L_{n+1}}$ -independent.

By (2.4), L_{n+1} is dense in $L_{n+1}\langle b \rangle$ for each $b \in B_{n+1}$. On the other hand if $b \in B_1 \cup \dots \cup B_n$, then L_{n+1} is neither dense nor tame in $L_{n+1}\langle b \rangle$. By (1.18), $B_1 \cup \dots \cup B_n \cup B_{n+1}$ is $\ll_{L_{n+1}}$ -independent. Again by (1.16), $B_1 \cup \dots \cup B_n \cup B_{n+1}$ is \ll_R -independent. \square

3. THE COMPLETION OF AN O-MINIMAL STRUCTURE.

(3.1) PROPOSITION. *Let T be an o-minimal expansion of the theory of real closed fields. Let $R \prec \mathcal{M}$ be models of T . Then there is a model S of T with $R \prec S \prec \mathcal{M}$, such that:*

(i) *R is dense in S .*

(ii) *If R' is an elementary restriction of \mathcal{M} , $R \prec R'$ and if R is dense in R' , then there is an elementary embedding $R' \rightarrow S$ over R .*

The embedding in (ii) is unique. If $R \prec S' \prec \mathcal{M}$ and S' has properties (i) and (ii), then there is a unique R -isomorphism $S \rightarrow S'$.

PROOF. Let $X \subseteq \mathcal{M}$ be the set of all $\alpha \in \mathcal{M}$, such that R is dense in $R\langle \alpha \rangle$. Let B be a \ll_R -basis of X over R . We claim, that $S := R\langle B \rangle$ has the required properties.

By (1.18), R is dense in S . Let $R' \prec \mathcal{M}$ be an elementary extension of R , such that R is dense in R' . Let B' be a transcendence basis of R' over R . Clearly B' is an \ll_R -basis of R' . By the choice of B , the type of every $b' \in B'$ over R is realized in $R\langle B \rangle$. By (1.8) we know that $tp(B'/R)$ is realized in $R\langle B \rangle = S$. Hence $tp(R'/R)$ is realized in S and there is an elementary R -embedding $R' \rightarrow S$.

Both additions are obvious. \square

(3.2) COROLLARY. *Let T be an o-minimal expansion of the theory of real closed fields. Let $R \prec \tilde{R} \prec \mathcal{M}$ be models of T , suppose that R is archimedean in \tilde{R} and \tilde{R} is tame in \mathcal{M} . We provide \tilde{R} with the topology induced by the ordering of \tilde{R} . Let S be the topological closure $\overline{\tilde{R}}$ in this topology. Then $R \prec S \prec \tilde{R} \prec \mathcal{M}$ and S fulfills the conditions (i) and (ii) of (3.1), both for R and \tilde{R} as well as for R and \mathcal{M} . We have*

$$S = \{\alpha \in \tilde{R} \mid R \text{ is dense in } R\langle\alpha\rangle\}$$

PROOF. Let $R \prec S_1 \prec \mathcal{M}$ as in (3.1) and let $S_1 \prec \tilde{S}_1 \prec \mathcal{M}$, such that S_1 is archimedean in \tilde{S}_1 and \tilde{S}_1 is tame in \mathcal{M} . Since \tilde{S}_1 and \tilde{R} are isomorphic over R we can suppose that $S_1 \subseteq \tilde{R} = \tilde{S}_1$ (T is an expansion of RCF). Since R is archimedean in \tilde{R} , S_1 is contained in $\overline{\tilde{R}} = S$. If $\alpha \in \overline{\tilde{R}}$, then R is dense in $R\langle\alpha\rangle$. If $\alpha \in \tilde{R}$, such that R is dense in $R\langle\alpha\rangle$, then by (1.17), the set S_1 is dense in $S_1\langle\alpha\rangle$. By the choice of S_1 we get therefore $\alpha \in S_1$. This proves $S_1 = S = \{\alpha \in \tilde{R} \mid R \text{ is dense in } R\langle\alpha\rangle\}$. \square

(3.1) applied to a sufficiently large, elementary extension \mathfrak{M} of R yields

(3.3) COROLLARY. *Let T be an o-minimal expansion of the theory of real closed fields and let R be a model of T . Then there is a model $S \succ R$ with:*

- (i) R is dense in S .
- (ii) If R' is an elementary extension of R and R is dense in R' , then there is an elementary embedding $R' \rightarrow S$ over R .

The embedding in (ii) is unique. S is uniquely determined up to a unique R -isomorphism by conditions (i) and (ii). \square

The model S in (3.3) is the largest elementary extension of R , such that R is dense in S . No type over S is dense. S is called the **completion** of R and is denoted by \hat{R} .

We get S by (3.2) in the following manner: choose $R \prec R_1 \prec \mathfrak{M}$ such that R is archimedean in R_1 , R_1 is tame in \mathfrak{M} and \mathfrak{M} is $|R_1|^+$ -saturated. Take

$$S = \{\alpha \in R_1 \mid R \text{ is dense in } R\langle\alpha\rangle\}$$

Since R is dense in $R\langle\alpha\rangle$ if and only if R is dense in the field $R(\alpha)$ (by (2.4)), the underlying field of the completion of R does not depend on the theory T .

If V is a convex valuation ring of R and \hat{V} is the convex hull of V in \hat{R} , then the valued field (\hat{R}, \hat{V}) is the completion of the valued field (R, V) .

4. DEFINITION OF THE PSEUDO COMPLETION

(4.1) THEOREM. *Let T be an o-minimal expansion of fields in the language \mathcal{L} , let $R \models T$ and let \mathcal{V} be a set of T -convex valuation rings of R (the case $R \in \mathcal{V}$ is not excluded). For each $V \in \mathcal{V}$ let $K_V, L_V \subseteq V$ be maximal subfields of V .*

Let $S \succ R$ so that S contains completions \hat{K}_V of K_V and \hat{L}_V of L_V for all $V \in \mathcal{V}$. Then

- (i) *There is an \mathcal{L} -isomorphism $\varphi : R\langle\bigcup_{V \in \mathcal{V}} \hat{K}_V\rangle \rightarrow R\langle\bigcup_{V \in \mathcal{V}} \hat{L}_V\rangle$ over R sending $R\langle\hat{K}_V\rangle$ onto $R\langle\hat{L}_V\rangle$.*
- (ii) *If $K_V = L_V$ for each $V \in \mathcal{V}$ and φ_V denotes the unique \mathcal{L} -isomorphism $\varphi_V : \hat{K}_V \rightarrow \hat{L}_V$ over K_V , then there is a unique \mathcal{L} -isomorphism $\varphi : R\langle\bigcup_{V \in \mathcal{V}} \hat{K}_V\rangle \rightarrow R\langle\bigcup_{V \in \mathcal{V}} \hat{L}_V\rangle$ over R extending all the φ_V .*

(iii) *The product map*

$$\bigotimes_{V \in \mathcal{V}} (R \otimes_{K_V} \hat{K}_V) \longrightarrow S$$

is injective.

PROOF. Let $B_V \subseteq \hat{K}_V$ be a basis of \hat{K}_V over K_V in the sense of T . Since K_V is dense in \hat{K}_V , B_V is \ll_{K_V} -independent. By (2.6), $B := \bigcup_{V \in \mathcal{V}} B_V$ is \ll_R -independent. Moreover by (2.4), for $V \in \mathcal{V}$ and $b \in B_V$, the cut of b over R is realized by some $c_b \in \hat{L}_V$ and in the situation of (ii) we must take $c_b := \varphi_V(b)$. Then, by (1.8), there is an elementary R -embedding $\varphi : R\langle B \rangle \longrightarrow R\langle \bigcup_{V \in \mathcal{V}} \hat{L}_V \rangle$ sending b to c_b for each $b \in B_V$, $V \in \mathcal{V}$.

In order to prove that φ is surjective and that $\varphi(R\langle \hat{K}_V \rangle) = R\langle \hat{L}_V \rangle$ it is enough to show that $C_V := \{c_b \mid b \in B_V\}$ is a basis of \hat{L}_V over L_V in the sense of T . Clearly C_V is independent over L_V . Let $l \in \hat{L}_V \setminus L_V$, $l \notin C_V$. Then l is a V -limit, so $tp(l/K_V)$ is realized in \hat{K}_V . Then also $tp(l/R)$ is realized in $R\langle B_V \rangle$, hence $tp(l/R)$ is realized in $R\langle C_V \rangle$. This means that $C_V \cup \{l\}$ is \ll_R -dependent and by (1.16), $C_V \cup \{l\}$ is \ll_{L_V} -dependent. Since $l \in \hat{L}_V$ and L_V is dense in \hat{L}_V this is only possible if $l \in L_V\langle C_V \rangle$.

This proves (i) and (ii).

(iii). First we show that $R \otimes_{K_V} \hat{K}_V \longrightarrow S$ is injective, i.e. R and \hat{K}_V are linearly disjoint over K_V . Since B_V is \ll_{K_V} -independent, B_V is \ll_R -independent by (2.6). Since $tp(b/K_V)$ is omitted in R for all $b \in B_V$ it follows that B_V is $\ll_{K_V, R}$ -independent. By (1.14), $tp(B_V/R)$ is an heir of $tp(B_V/K_V)$. This property implies that every linear equation with coefficients in \hat{K}_V which has a solution in R , also has a solution in K_V . Hence R and \hat{K}_V are linearly disjoint over K_V .

It remains to show that the domains $R \otimes_{K_V} \hat{K}_V$ are linearly disjoint over R . By what we have shown we may identify $R \otimes_{K_V} \hat{K}_V$ with $R[\hat{K}_V] \subseteq S$. Moreover we may assume that \mathcal{V} is finite, say $\mathcal{V} = \{V_1, \dots, V_n\}$. We write B_i for B_{V_i} . Since $\bigcup_{i=1}^n B_i$ is \ll_R -independent, the type $tp(B_n/R \cup B_1 \cup \dots \cup B_{n-1})$ is an heir over R (c.f. (1.14)). Again it follows that every linear equations with coefficients in $R\langle B_1 \cup \dots \cup B_{n-1} \rangle$ which has a solution in $R\langle B_n \rangle$, also has a solution in R . By induction on n we get (iii). \square

(4.2) DEFINITION. In the situation of (4.1) the model $R\langle \bigcup_{V \in \mathcal{V}} \hat{K}_V \rangle$ of T is called the **pseudo completion** of R with respect to \mathcal{V} .

By Theorem (4.1) this model of T is up to an R -isomorphism independent of S , K_V and \hat{K}_V ; it can be constructed in the following way. Let A be the ring

$$A := \bigotimes_{V \in \mathcal{V}} (R \otimes_{K_V} \hat{K}_V).$$

Then A is an R -algebra without zero divisors and there is an injective R -algebra homomorphism f from A into an elementary extension of R . Then the pseudo completion is the definable closure of $f(A)$.

If T is the theory of real closed fields then the pseudo completion is the real closure of the quotient field of A with respect to any ordering.

The next proposition describes in which sense, the pseudo completion is minimal.

(4.3) PROPOSITION. *Let T be an o -minimal expansion of fields in the language \mathcal{L} , let $R \models T$, let \mathcal{V} be a family of T -convex subrings of R and let R' be the pseudo completion of R with respect to \mathcal{V} . Let $S \succ R$ be an elementary extension of R .*

(i) *Suppose each cut of R , which is a V -limit for some $V \in \mathcal{V}$ is realized in S . Then there is an elementary embedding $R' \longrightarrow S$ over R .*

(ii) For each $V \in \mathcal{V}$, let $W(V)$ be the convex hull of V in S . Let \mathcal{W} be a set of T -convex valuation rings of S with $W(V) \in \mathcal{W}$ for all $V \in \mathcal{V}$ and let S' be the pseudo completion of S with respect to \mathcal{W} . Then there is an elementary R -embedding $R' \rightarrow S'$.

If \mathcal{W} is precisely the set of all $W(V)$ with $V \in \mathcal{V}$ and for each $V \in \mathcal{V}$, the residue field of V is equal to the residue field of $W(V)$, then we can choose this embedding φ so that S' is the definable closure of $S \cup \varphi(R')$.

PROOF. For $V \in \mathcal{V}$ let $K_V \subseteq V$ be a maximal subfield of V .

(i). Let $B_V \subseteq \hat{K}_V \subseteq R'$ be a transcendence basis of \hat{K}_V over K_V ($V \in \mathcal{V}$). Pick some $V \in \mathcal{V}$. By assumption and (2.4), for $b \in B_V$ the cut of b over K_V is realized in S . Since K_V is dense in \hat{K}_V , B_V is \ll_{K_V} -independent. By (1.8), \hat{K}_V can be embedded into S over K_V . By (4.1), R' can be embedded over R into S .

(ii). Let $L_V \subseteq W(V)$ be a maximal subfield of $W(V)$ containing K_V for every $V \in \mathcal{V}$. Since K_V is archimedean in L_V it follows from (1.17)(ii), that there is a (unique) elementary K_V -embedding $\varphi_V : \hat{K}_V \rightarrow \hat{L}_V$. By (4.1) we may assume that S' contains the definable closure of $S[\bigcup_{V \in \mathcal{V}} \hat{L}_V]$. By (4.1), R' is R -isomorphic to the definable closure of $R[\bigcup_{V \in \mathcal{V}} \varphi(\hat{K}_V)]$ in S' .

Now suppose \mathcal{W} is precisely the set of all $W(V)$ with $V \in \mathcal{V}$ and for each $V \in \mathcal{V}$, the residue field of V is equal to the residue field of $W(V)$. Then $L_V = K_V$ and S' is R -isomorphic to the definable closure of $S[\bigcup_{V \in \mathcal{V}} \hat{K}_V]$. \square

By (5.11) below, a pseudo completion R' of a pure real closed field R is in general not minimal in the sense that any R -endomorphism of R' is an automorphism. Moreover it is unclear if R' is uniquely determined up to an R -isomorphism by the minimality demand of (4.3)(i); this is the content of the open problem (5.12) at the end of the paper.

5. COMPLETION IN STAGES OF POLYNOMIALLY BOUNDED STRUCTURES.

An o-minimal expansion R of a field is called polynomially bounded if every definable function $R \rightarrow R$ is ultimately bounded by some polynomial. Here all polynomially bounded structures are additionally assumed to have an archimedean prime model. In particular, real closed fields are polynomially bounded. If R is polynomially bounded, then every convex subring is $Th(R)$ -convex (c.f. [vdD-Lew]).

(5.1) DEFINITION. Let K be an ordered field and let \mathcal{V} be a set of convex valuation rings of K . We say that K is complete in stages with respect to \mathcal{V} if all residue fields of elements of \mathcal{V} are complete.

By (2.4), K is complete in stages with respect to \mathcal{V} if and only if there are no V -limits in any ordered field extension of K , for all $V \in \mathcal{V}$.

For ordered field, this definition is more general than the definition of Ribenboim, [Ri]. Let V be a convex valuation ring of an ordered field. Then the valued field (K, V) is complete in stages in the sense of Ribenboim ([Ri], section D.) if and only if K is complete in stages with respect to

$$\{W \subseteq K \mid W \text{ is a convex valuation ring with } V \subsetneq W\}$$

in our sense. This follows from (2.5) together with [Ri], section D., Théorème 3, which says that the valued field (K, V) is complete in stages if and only if every distinguished pseudo Cauchy sequence of (K, V) has a pseudo limit in K .

In this section we construct a completion in stages of R with respect to \mathcal{V} for a polynomially bounded expansion R of a real closed field and a set \mathcal{V} of convex valuation rings of R . This is a smallest elementary extension S which is complete in stages with respect to the set of convex hulls of the $V \in \mathcal{V}$. We get S by iterating the construction of the pseudo completion. Before we can do this, we have to compute the residue fields and the value groups of the pseudo completion.

(5.2) PROPOSITION. *Let R be polynomially bounded and let s be an element from an elementary extension of R , $s \notin R$. The following are equivalent.*

- (i) $\text{sign}(s/R) = 0$.
- (ii) *If G is a convex subgroup of $(R, +)$, then G^+ is omitted in $R\langle s \rangle$.*
- (iii) *If W is a convex valuation ring of $R\langle s \rangle$, then the value group of W is equal to the value group of $W \cap R$.*

PROOF. Clearly (ii) implies (i). Also (ii) implies (iii), since an element $R\langle s \rangle$ which is not in the value group of $W \cap R$ is the edge of a convex subgroup of R .

Conversely suppose $\alpha \in R\langle s \rangle$ realizes G^+ for a convex subgroup G of $(R, +)$. The proposition is proved if we show that $\text{sign}(s/R) \neq 0$ and that $w(\alpha)$ is not in the value group of $V(\alpha/R)$, where w is the valuation of $R\langle s \rangle$ with respect to the convex hull W of $V(\alpha/R)$ in $R\langle s \rangle$.

In order to see this, let $r \in R$ and suppose $\alpha/r \in W^*$, say $\alpha/r > 0$. Then there are $y, z \in V$ with $0 < \alpha/r < y$ and $0 < r/\alpha < z$, thus $0 < r/z < \alpha < y \cdot r$. Hence $r/z \in G$ and $zy \cdot r/z \notin G$ in contradiction to $z \cdot y \in V(\alpha/R)$.

Hence $w(\alpha)$ is not in the value group of $V(\alpha/R)$. By the valuation property ([vdD-S]) there must be some $b \in R$ such that $w(s - b)$ is not in the value group of $V(\alpha/R)$. But then $s - b$ realizes the edge of a convex subgroup of R , i.e. $\text{sign}(s/R) \neq 0$. \square

(5.3) LEMMA. *Let R be polynomially bounded and let s be from an elementary extension of R with $\text{sign}(s/R) = 0$. If $F : R \rightarrow R$ is R -definable with $F(s) \notin R$, then there are $a, b \in R$, $a < s < b$ such that F is differentiable in (a, b) and for all $r \in R$ with $a < r < b$ we have*

$$G(F(s)/R) = F'(r) \cdot G(s/R).$$

PROOF. By C^1 -cell decomposition and since the cut of s over R is not definable, we may assume that F is C^1 in an open neighborhood of $[a, b]$ for some $a, b \in R$ with $a < s < b$. We write F' for the derivative of F in $[a, b]$. If F is a linear map in some interval (c, d) with $c, d \in R$, $c < s < d$ the lemma holds since $G(ys + z/R) = yG(s/R)$ for all $y, z \in R$, $y \neq 0$. Hence we may assume that $F'(s) \notin R$.

Let W be the convex hull of V in $R\langle s \rangle$. Since $\text{sign}(s/R) = 0$, (5.2) implies that the value group of W is equal to the value group of V . Hence there is some $z \in R$ such that $z \cdot F'(s) \in W^*$. We may replace F by $z \cdot F$, hence we may assume that $F'(s) \in W^*$, say $F'(s) > 0$. Since $F'(s) \notin R$, (5.2) gives us $c, d \in V$, $\mathfrak{m}_V < c < d$ with $c < F'(s) < d$. By shrinking (a, b) if necessary we may assume that $F|_{[a, b]} : [a, b] \rightarrow [F(a), F(b)]$ is a strictly increasing homeomorphism with $F'(x) \in (c, d)$ on $[a, b]$. We prove $G(F(s)/R) = G(s/R)$; this also proves the lemma, since $G(s/R) = F'(r) \cdot G(s/R)$ for all $r \in R$, $a < r < b$.

In order to show $G(s/R) \subseteq G(F(s)/R)$ we take $g \in G(s/R)$, $g > 0$, $r \in R$ with $a < r < s$ and we show that $F(r) + g < F(s)$. Since $F'(x) > c$ in $[a, b]$ we know that $F(x) > F(r) + c \cdot (x - r)$ for $x \in (r, b)$. Since $g \in G(s/R)$ and $c \in V^*$, we know that $r + g/c < s$, hence $F(x) > F(r) + c \cdot (x - r) \geq F(r) + g$ for $x \in (r + g/c, b)$ and $F(s) > F(r) + g$ as desired.

Conversely let $y \in R$ with $y > G(s/R)$. Then also $y/d > G(s/R)$ and there is some $r \in (a, b)$ with $r < s < r + y/d$. Since $F'(x) < d$ in $[a, b]$ we know that $F(x) < F(r) + d(x - r)$ for $x \in (r, b)$. Hence also $F(x) < F(r) + d \cdot (x - r) < F(r) + y$ for all $x \in R$ with $r < x < \min\{b, r + y/d\}$. Since $r < s < \min\{b, r + y/d\}$ it follows $F(r) < F(s) < F(r) + y$, thus $y \notin G(F(s)/R)$ as desired. \square

(5.4) LEMMA. *Let $R \prec S$ be polynomially bounded, such that $\text{sign } s/R = 0$ for all $s \in S \setminus R$. Let α be from an elementary extension of S and let $F : S \rightarrow S$ be S -definable such that $F(\alpha) \notin S$. Suppose $\text{sign}(\alpha/S) = 0$ and the cuts of α and $F(\alpha)$ over R are omitted in S . Then there is some $c \in R^*$ such that $G(F(\alpha)/R) = c \cdot G(\alpha/R)$.*

PROOF. By (5.3) applied to S and α we get some $s \in S^*$ with $G(F(\alpha)/S) = s \cdot G(\alpha/S)$. Let $V = V(\alpha/R)$ and let W be the convex hull of V in S . By assumption and by (5.2), there is some $c \in R^*$ such that $c/s \in W^*$. Since the cut of α over R is omitted in S , $G(\alpha/S)$ contains $G(\alpha/R)$. Since S does not realize the upper edge of $G(\alpha/R)$, $G(\alpha/S)$ is the convex hull of $G(\alpha/R)$ in S . This implies that $V(\alpha/S)$ contains $V = V(\alpha/R)$. Again, since S does not realize the upper edge of V , $V(\alpha/S)$ is the convex hull of V in S . Thus c/s is a unit in $V(\alpha/S) = W$ and $s \cdot G(\alpha/S) = s \cdot (c/s) \cdot G(\alpha/S) = c \cdot G(\alpha/S)$.

By (5.2), $\text{sign } F(\alpha)/S = 0$, hence also $G(F(\alpha)/S)$ is the convex hull of $G(F(\alpha)/R)$ in S . Thus $G(F(\alpha)/S) = c \cdot G(\alpha/S)$ implies $G(F(\alpha)/R) = c \cdot G(\alpha/R)$. \square

(5.5) COROLLARY. *Let R be polynomially bounded, let s_1, \dots, s_n be from an elementary extension of R with $\text{rk}_R(s_1, \dots, s_n) = n$ and $\text{sign}(s_i/R) = 0$ for all $i \in \{1, \dots, n\}$. If $F : R^n \rightarrow R$ is R -definable and $F(s_1, \dots, s_n) \notin R$, then $\text{sign}(F(s_1, \dots, s_n)/R) = 0$ and $G(F(s_1, \dots, s_n)/R) = c \cdot G(s_i/R)$ for some $i \in \{1, \dots, n\}$ and some $c \in R$.*

PROOF. For $i \in \{1, \dots, n\}$ the cut of s_i over R is omitted in $R\langle s_1, \dots, s_{i-1} \rangle$. As $\text{sign}(s_i/R) = 0$ it follows that $\text{sign}(s_i/R\langle s_1, \dots, s_{i-1} \rangle) = 0$. Then by induction on n , (5.2) implies that every $s \in R\langle s_1, \dots, s_i \rangle \setminus R$ has signature 0.

Let $\alpha := F(s_1, \dots, s_n)$. Since $\alpha \notin R$ there is some $i \in \{1, \dots, n\}$ such that $\alpha \cup (\{s_1, \dots, s_n\} \setminus \{s_i\})$ is a \ll_R -basis of $R\langle s_1, \dots, s_n \rangle$. Say $i = 1$. Let $S := R\langle s_2, \dots, s_n \rangle$. Since the cuts of α and s_1 over R are omitted in S and $\text{sign}(s_1/S) = 0$, we can apply (5.4). Hence $G(\alpha/R) = c \cdot G(s_1/R)$ for some $c \in R^*$. \square

Now we compute the residue fields and the value groups of convex valuation rings of the pseudo completion of a polynomially bounded structure:

(5.6) THEOREM. *Let R be polynomially bounded and let S be the pseudo completion of R with respect to a set \mathcal{V} of convex subrings of R .*

- (i) *Every $s \in S \setminus R$ is a V -limit for a unique convex valuation ring V of R and this ring is in \mathcal{V} .*
- (ii) *Let V_0 be any convex valuation ring of R . Then the convex hull W_0 of V_0 in S is the unique convex valuation ring of S lying over V_0 . The value group of W_0 is the value group of V_0 and*
 - a. *if $V_0 \subsetneq V$ for all $V \in \mathcal{V}$, then the extension $(R, V_0) \subseteq (S, W_0)$ of valued fields is immediate;*
 - b. *if $V \subseteq V_0$ for some $V \in \mathcal{V}$, then W_0/\mathfrak{m}_{W_0} is the pseudo completion of V_0/\mathfrak{m}_{V_0} with respect to $\{V/\mathfrak{m}_V \mid V \in \mathcal{V}, V \subseteq V_0\}$.*

PROOF. (i) follows from (5.5), since S is the definable closure of a \ll_R -independent set of elements, each being a V -limit for some $V \in \mathcal{V}$ (c.f. (2.6) and (2.4)). The uniqueness statement holds by (2.4).

(ii). By (i), every $s \in S \setminus R$ is a V -limit for some $V \in \mathcal{V}$, in particular $\text{sign}(s/R) = 0$. By (5.2), no edges of convex subgroups of R are realized in S . Consequently W_0 is the unique convex valuation ring of S , lying over V_0 and W_0 must have the same value group as V_0 .

In order to see a. and b. let $K_V \subseteq V$ be a maximal subfield for each $V \in \mathcal{V} \cup \{V_0\}$.

a. Suppose $V_0 \subsetneq V$ for all $V \in \mathcal{V}$. Let $s \in W_0$. We have to show that the cut p of s over K_{V_0} is definable. Suppose p is not definable. If $G(s/K_{V_0}) = 0$, then s is a V_0 -limit by (2.4). By (i), s is a V -limit for some $V \in \mathcal{V}$. Since $V \neq V_0$ this is impossible (c.f. (2.4)). Hence $G(s/K_{V_0}) \neq 0$ and $V(s/K_{V_0})$ is a proper convex valuation ring of K_{V_0} . Since p is omitted in R , $G(s/R)$ is the largest convex subgroup of R with $G(s/R) \cap K_{V_0} = G(s/K_{V_0})$. This implies that $V(s/R)$ is a convex valuation ring, lying over $V(s/K_{V_0})$. As $V(s/K_{V_0})$ is proper it follows $V(s/R) \subseteq V_0$. On the other hand - by (i) - $V(s/R) \in \mathcal{V}$ and this contradicts our assumption on V_0 .

b. By (4.1) we may assume that $K_{V_0} \subseteq K_V$ for every $V \in \mathcal{V}$ with $V_0 \subseteq V$. For $V \in \mathcal{V}$ with $V \subseteq V_0$, any maximal subfield of $V \cap K_{V_0}$ is also a maximal subfield of V (this is so, since such a field L is archimedean in $V \cap K_{V_0}$ and tame in K_{V_0} - as $V \cap K_{V_0}$ is archimedean in V and K_{V_0} is tame in R , also L is archimedean in V and tame in R). So by (4.1) we may assume that $K_V \subseteq K_{V_0}$ for all $V \in \mathcal{V}$ with $V \subseteq V_0$, too. Let $\mathcal{V}' := \{V \in \mathcal{V} \mid V \subseteq V_0\}$ and let $R' := R(\bigcup_{V \in \mathcal{V}'} \hat{K}_V)$ be the pseudo completion of R with respect to \mathcal{V}' .

First we prove (ii) b. for \mathcal{V}' and $W'_0 := W_0 \cap R'$. By (4.1) it is enough to show that $K_{V_0}(\bigcup_{V \in \mathcal{V}'} \hat{K}_V)$ is a maximal subfield of W'_0 . In order to prove this it suffices to take $V_1, \dots, V_n \in \mathcal{V}'$ and finite subsets $B_i \subseteq K_{V_i}$ independent over K_{V_i} ($1 \leq i \leq n$) and to show that $K_{V_0}\langle B \rangle$ is a maximal subfield of $W_0 \cap R\langle B \rangle$. By (2.6), the B_i are mutually disjoint and their union B is $\ll_{K_{V_0}}$ -independent. Hence for each $b \in B$ the cut p of b over K_{V_0} is omitted in $K_{V_0}\langle B \setminus \{b\} \rangle$. Since p is not definable, the unique extension on $K_{V_0}\langle B \setminus \{b\} \rangle$ is not definable as well. This shows that $K_{V_0}\langle B \setminus \{b\} \rangle$ is archimedean in $K_{V_0}\langle B \rangle$. Hence, by induction, K_{V_0} is archimedean in $K_{V_0}\langle B \rangle$ and $K_{V_0}\langle B \rangle$ is a subfield of W_0 . Since $R\langle B \rangle$ is generated by B and $\dim R\langle B \rangle/R$ is greater or equal to the dimension of the residue field of $W_0 \cap R\langle B \rangle$ over V_0/\mathfrak{m}_{V_0} (c.f. [vdD-Lew]), $K_{V_0}\langle B \rangle$ must be a maximal subfield of $W_0 \cap R\langle B \rangle$.

Hence we know that the residue field of $W'_0 = W_0 \cap R'$ is the pseudo completion of V_0/\mathfrak{m}_{V_0} with respect to $\{V/\mathfrak{m}_V \mid V \in \mathcal{V}, V \subseteq V_0\}$ and it remains to show that $(R', W'_0) \subseteq (S, W_0)$ is immediate. But this follows from a., since by (4.1), S is the pseudo completion of R' with respect to the set of convex hulls of all $V \in \mathcal{V}$ with $V_0 \subsetneq V$. \square

(5.7) EXAMPLE. *The pseudo completion R' of a real closed field R with respect to a set \mathcal{V} of convex valuation rings of R , containing R , is not complete in general. In particular, if \mathcal{V}' denotes the set of convex hulls of elements from \mathcal{V} in R' , then R' need not be complete in stages with respect to \mathcal{V}' .*

To see an example, let K be a real closed field with completion $\hat{K} \neq K$ and let Γ be a divisible subgroup of $(\mathbb{R}, +)$ containing $1 \in \mathbb{R}$. We also assume that Γ is an ordered subgroup of $(K, +)$. In this situation we can equip the generalized power series field $\hat{K}((t^\Gamma))$ with the derivative

$$\left(\sum a_\gamma t^\gamma\right)' = \sum a_\gamma \cdot \gamma \cdot t^{\gamma-1}.$$

Let R be the real closure of $K(t^\gamma \mid \gamma \in \Gamma)$ in $\hat{K}((t^\Gamma))$ and let V be the convex hull of K in R . The completion \hat{R} of R is

$$\hat{R} = \left\{ \sum_{n=0}^{\infty} a_n t^{\gamma_n} \mid a_n \in K, \gamma_n \in \Gamma \text{ and } \gamma_n \rightarrow \infty (n \rightarrow \infty) \right\}.$$

Hence the pseudo completion of R with respect to $\{V, R\}$ is $\hat{R}\langle\hat{K}\rangle$. We claim that for $x \in \hat{K} \setminus K$, the element

$$\exp(x \cdot t) := \sum_{i=0}^{\infty} \frac{x^i}{i!} t^i$$

is not in $\hat{R}\langle\hat{K}\rangle$. Since $\exp(x \cdot t)$ is in the completion of $\hat{K}(t^\gamma \mid \gamma \in \Gamma) \subseteq \hat{R}\langle\hat{K}\rangle$, this will show the incompleteness of $\hat{R}\langle\hat{K}\rangle$.

We use a differential algebraic argument.

(5.8) LEMMA. *Let $K \subseteq L$ be ordinary differential fields of characteristic 0, let $y, x \in L$, $y \neq 0$ such that x is transcendental over K . Suppose $g, h \in K[x]$, with $y' = g \cdot y$ and $x' = h$. If $g \notin K$ and $\deg g \geq \deg h$, then y and x are algebraically independent over K . Here the degree is the degree with respect to x .*

PROOF. Suppose y is algebraic over $K(x)$. Let $f_{d-1}, \dots, f_0 \in K(x)$ be rational functions, such that

$$\mu(T) := T^d + f_{d-1}T^{d-1} + \dots + f_0$$

is the minimal polynomial of y over $K(x)$. Then

$$\begin{aligned} 0 &= \mu(y)' = d \cdot y^{d-1} \cdot y' + f'_{d-1} y^{d-1} + f_{d-1} \cdot (d-1) \cdot y^{d-2} y' + \dots + f'_1 y + f_1 y' + f'_0 = \\ &= d \cdot g \cdot y^d + (f'_{d-1} + f_{d-1} \cdot (d-1) \cdot g) \cdot y^{d-1} + \dots + (f'_1 + f_1 g) y + f'_0 =: \eta(y). \end{aligned}$$

Since $x' \in K(x)$, $K(x)$ is a differential subfield of L and $\eta(y) = 0$ is an algebraic relation of y over $K(x)$ of degree d . Hence $\eta(y) = d \cdot g \cdot \mu(y)$ and a comparison of the constant coefficients with respect to y implies $f'_0 = d \cdot g \cdot f_0$. Let $P, Q \in K[T]$ with $f_0 = P(x)/Q(x)$, $Q(x) \neq 0$. Since $y \neq 0$, $P(x) \neq 0$. From $f'_0 = d \cdot g \cdot f_0$ we get

$$Q(x) \cdot P(x)' - P(x) \cdot Q(x)' = d \cdot g \cdot P(x) \cdot Q(x).$$

Since $x' = h$ and $\deg g \geq \max\{1, \deg h\}$, $P(x)'$ is a polynomial in x of degree $< \deg P + \deg g$. Also $\deg Q(x)' < \deg Q + \deg g$, hence $\deg(Q(x) \cdot P(x)' - P(x) \cdot Q(x)') < \deg P + \deg Q + \deg g = \deg(d \cdot g \cdot P(x) \cdot Q(x))$, a contradiction. \square

Now we prove $\exp(x \cdot t) \notin \hat{R}\langle\hat{K}\rangle$. Let $B \subseteq \hat{K}$ be a transcendence basis of \hat{K} over K containing x and let C be a transcendence basis of \hat{R} over R . By (2.6), $B \cup C$ is a transcendence basis of $\hat{R}\langle\hat{K}\rangle$ over R and $B \cap C = \emptyset$. Let $L := \hat{R}\langle B \setminus \{x\} \rangle$. The field \hat{R} is a differential subfield of $\hat{K}((t^\Gamma))$, equipped with the derivative introduced above (thus $(t^\gamma)' = \gamma \cdot t^{\gamma-1}$ for $\gamma \in \Gamma$). Since L is obtained from \hat{R} by adjoining constants to \hat{R} and then taking the real closure, L is also a differential subfield of $\hat{K}((t^\Gamma))$. Moreover x is transcendental over L . Since $\exp(x \cdot t)' = x \cdot \exp(x \cdot t)$ and $x' = 0$, (5.8) implies that $\exp(x \cdot t)$ and x are algebraically independent over L . Hence $\exp(x \cdot t) \notin L(x) = \hat{R}\langle\hat{K}\rangle$ as desired. \square

In the example above, the pseudo completion of $S := \hat{R}(\hat{K})$ with respect to $\{S, W\}$, where W is the convex hull of V , is the completion of $\hat{K}(t^\gamma \mid \gamma \in \Gamma)$.

More generally, if R is polynomially bounded, \mathcal{V} is a set of convex valuation rings of R , let R' be the pseudo completion of R with respect to \mathcal{V} and let \mathcal{V}' be the set of convex hulls of elements from \mathcal{V} in R' . We write $(R, \mathcal{V})'$ for (R', \mathcal{V}') . We define for each ordinal α the pair $(R^{(\alpha)}, \mathcal{V}^{(\alpha)})$ by $(R^{(0)}, \mathcal{V}^{(0)}) := (R, \mathcal{V})$, $(R^{(\alpha+1)}, \mathcal{V}^{(\alpha+1)}) := (R^{(\alpha)}, \mathcal{V}^{(\alpha)})'$ and for a limit ordinal α we take $R^{(\alpha)} = \bigcup_{\beta < \alpha} R^{(\beta)}$ and $\mathcal{V}^{(\alpha)} := \{\bigcup_{\beta < \alpha} V^{(\beta)} \mid V \in \mathcal{V}\}$.

Let $V_0 \subseteq \bigcap_{V \in \mathcal{V}} V$ be a convex subring and let $V_0^{(\alpha)}$ be the convex hull of V_0 in $R^{(\alpha)}$.

Claim. The extension $(R^{(1)}, V_0^{(1)}) \subseteq (R^{(\alpha)}, V_0^{(\alpha)})$ is immediate for all $\alpha \geq 1$.

PROOF. By induction on α , where the limit step is obvious. Suppose we know that $(R^{(1)}, V_0^{(1)}) \subseteq (R^{(\alpha)}, V_0^{(\alpha)})$ is immediate for some $\alpha \geq 1$. We show that $(R^{(\alpha)}, V_0^{(\alpha)}) \subseteq (R^{(\alpha+1)}, V_0^{(\alpha+1)})$ is immediate. If $V_0 \notin \mathcal{V}$, then $V_0^{(\alpha)} \notin \mathcal{V}^{(\alpha)}$ and we can apply (5.6)(ii)a.

Hence we may assume that V_0 is the least element in V . Then also $V_0^{(\alpha)}$ is the least element of $\mathcal{V}^{(\alpha)}$. By (5.6)(ii)b., $V_0^{(1)}$ has a complete residue field. By induction, $V_0^{(\alpha)}$ has a complete residue field, too. Hence $R^{(\alpha+1)}$ is the pseudo completion of $R^{(\alpha+1)}$ with respect to $\mathcal{V}^{(\alpha)} \setminus \{V_0^{(\alpha)}\}$. But then again by (5.6)(ii)a., $(R^{(\alpha)}, V_0^{(\alpha)}) \subseteq (R^{(\alpha+1)}, V_0^{(\alpha+1)})$ is immediate. \square

From the claim it follows that $R^{(\alpha)}$ can be embedded as a field into the maximal immediate extension of the valued field $(R^{(1)}, V_0^{(1)})$. Consequently there must be some ordinal α with $R^{(\alpha)} = R^{(\alpha+1)}$.

(5.9) **DEFINITION.** The **completion in stages** of R with respect to \mathcal{V} is defined to be the elementary extension $R^{(\alpha)}$ for an ordinal α with $R^{(\alpha)} = R^{(\alpha+1)}$.

By construction, the completion in stages is complete in stages with respect to the family of convex hulls of the rings from \mathcal{V} . Moreover the properties of the pseudo completion from (5.6) are inherited by the completion in stages:

(5.10) **THEOREM.** *Let R be polynomially bounded and let S be the completion in stages of R with respect to a set \mathcal{V} of convex subrings of R .*

- (i) *Every $s \in S \setminus R$ is a V -limit for a unique convex valuation ring V of R and this ring is in \mathcal{V} .*
- (ii) *Let V_0 be any convex valuation ring of R . Then the convex hull W_0 of V_0 in S is the unique convex valuation ring of S lying over V_0 . The value group of W_0 is the value group of V_0 and*
 - a. *if $V_0 \subsetneq V$ for all $V \in \mathcal{V}$, then the extension $(R, V_0) \subseteq (S, W_0)$ of valued fields is immediate;*
 - b. *if $V \subseteq V_0$ for some $V \in \mathcal{V}$, then W_0/\mathfrak{m}_{W_0} is the completion in stages of V_0/\mathfrak{m}_{V_0} with respect to $\{V/\mathfrak{m}_{V_0} \mid V \in \mathcal{V}, V \subseteq V_0\}$.*
- (iii) *Let S' be an elementary extension of R and for each $V \in \mathcal{V}$ let W'_V be the convex hull of V in S' . If S' is complete in stages with respect to $\{W'_V \mid V \in \mathcal{V}\}$, then there is an elementary embedding $\varphi : S \rightarrow S'$ over R .*

PROOF. For an ordinal α , let $R^{(\alpha)}$ and $\mathcal{V}^{(\alpha)}$ be as in the construction of S above. First we prove (ii). Let V_0 be any convex valuation ring of R and let $V_0^{(\alpha)}$ be the convex hull of V_0

in $R^{(\alpha)}$. By induction on α we get from (5.6) that $V_0^{(\alpha)}$ is the unique convex valuation ring of $R^{(\alpha)}$, lying over V_0 and the value group of $V_0^{(\alpha)}$ is the value group of V_0 . Moreover item (ii)a. follows immediately from (5.6)(ii)a. by induction on α .

(ii)b. By (5.6)(ii)b. for every ordinal α , $V_0^{(\alpha+1)}/\mathfrak{m}_{V_0^{(\alpha+1)}}$ is the pseudo completion of $V_0^{(\alpha)}/\mathfrak{m}_{V_0^{(\alpha)}}$ with respect to $\{V^{(\alpha)}/\mathfrak{m}_{V^{(\alpha)}} \mid V \in \mathcal{V}, V \subseteq V_0\}$. By induction on α we get that $V_0^{(\alpha)}/\mathfrak{m}_{V_0^{(\alpha)}}$ is the α -fold iterated pseudo completion of V_0/\mathfrak{m}_{V_0} with respect to $\{V/\mathfrak{m}_V \mid V \in \mathcal{V}, V \subseteq V_0\}$. This easily implies (ii)b.

(i). The uniqueness statement is obviously true.

By induction on α we prove that every $x \in R^{(\alpha)} \setminus R$ is a V -limit for some $V \in \mathcal{V}$. For $\alpha = 1$ we know this from (5.6)(i). For limit ordinals there is nothing to do. Now suppose $x \in R^{(\alpha+1)}$. If the cut of x over R is realized in $R^{(\alpha)}$, then by the induction hypothesis, x is a V -limit for some $V \in \mathcal{V}$. Hence we may assume that the cut of x over R is omitted in $R^{(\alpha)}$. Since $R^{(\alpha+1)}$ is the pseudo completion of $R^{(\alpha)}$ with respect to $\mathcal{V}^{(\alpha)}$, (5.6)(i) gives us some $V \in \mathcal{V}$ such that x is a $V^{(\alpha)}$ -limit. Thus $\text{sign}(x/R^{(\alpha)}) = 0$ and for some $a \in R^{(\alpha)}$, $G(x/R^{(\alpha)}) = a \cdot \mathfrak{m}_{V^{(\alpha)}}$. Since the cut of x over R is omitted in $R^{(\alpha)}$, we have $\text{sign}(x/R) = 0$.

Since the value group of $V^{(\alpha)}$ is the value group of V , there is some $r \in R$ such that $r/a \in (V^{(\alpha)})^*$. Hence $a \cdot \mathfrak{m}_{V^{(\alpha)}} = r \cdot \mathfrak{m}_{V^{(\alpha)}}$. Since the cut of x over R is omitted in $R^{(\alpha)}$ and $G(x/R)^+$ is omitted in $R^{(\alpha)}$, $G(x/R^{(\alpha)}) = r \cdot \mathfrak{m}_{V^{(\alpha)}}$ is the convex hull of $G(x/R)$. Since $\mathfrak{m}_{V^{(\alpha)}}$ is the convex hull of \mathfrak{m}_V , it follows that $G(x/R) = r \cdot \mathfrak{m}_V$. Together with $\text{sign}(x/R) = 0$, this means that x is a V -limit.

(iii) Since W_V^+ is the convex hull of V and the residue field of W_V^+ is complete, for every maximal subfield K of V there is a completion of K inside W_V^+ . By (4.1), there is an elementary embedding of $R^{(1)}$ into S' over R . By an obvious induction this can be iterated until we reach the completion in stages. \square

If \mathcal{V} is finite of size n , then $R^{(n)}$ is complete in stages with respect to $\mathcal{V}^{(n)}$. This follows from (5.6) by induction on n : if $\mathcal{V} = \{V_1, \dots, V_n\}$ with $V_1 \subsetneq \dots \subsetneq V_n$, then by (5.6)(ii) b., $V_1^{(1)}$ has a complete residue field. Thus $R^{(2)}$ is the pseudo completion of $R^{(1)}$ with respect to $\{V_2^{(1)}, \dots, V_n^{(1)}\}$. Moreover $V_1^{(1)} \subseteq V_1^{(2)}$ is immediate by (5.6)(ii) a. Hence by induction, $R^{(n)}$ is complete in stages with respect to $\mathcal{V}^{(n)}$.

(5.11) EXAMPLE. *One might ask if the pseudo completion or the completion in stages S of a real closed field R with respect to a set of convex valuation rings is minimal in the sense that every R -embedding $S \rightarrow S$ is surjective. This is not true in general. Look at the following example.*

Let $R = R_0\langle\mu\rangle$ be the real closure of $\mathbf{Q}\langle\mu\rangle$, where μ is infinitesimal and let S be the pseudo completion of R with respect to the valuation ring $V :=$ the convex hull of \mathbf{Q} in R . Then S is $\mathbf{R}\langle\mu\rangle$, which is the completion in stages of R with respect to the valuation ring V , too. We now construct a proper real closed subfield R of $\mathbf{R}\langle\mu\rangle$, which contains μ and which is isomorphic over $R_0\langle\mu\rangle$ to S . In particular R realizes every cut of R_0 .

Let $T \subseteq \mathbf{R}$ be a transcendence basis over R_0 and let $B = \{b_1, b_2, \dots\}$ be a countable subset of T . Let

$$R := R_0\langle(T \setminus B) \cup \{\mu, b_1 + \mu b_2, b_2 + \mu b_3, \dots\}\rangle.$$

Then $b_1 \notin R$, otherwise there is some $n \in \mathbf{N}$ such that $b_1 \in R_1 := R_0\langle(T \setminus B) \cup \{\mu, b_1 + \mu b_2, \dots, b_n + \mu b_{n+1}\}\rangle$. But then $b_1, \dots, b_{n+1}, \mu \in R_1$, hence R_1 has transcendence degree $\geq n + 2$ over $R\langle T \setminus B \rangle$, which is not possible.

R is isomorphic to S over $R((T \setminus B) \cup \{\mu\})$, the isomorphism is given by sending b_i to $b_i + \mu \cdot b_{i+1}$ (observe that $T \cup \{\mu\}$ is \ll_{R_0} -independent and b_i and $b_i + \mu b_{i+1}$ realize the same cut over R_0 . Then use (1.8)).

(5.12) OPEN PROBLEM. Let S be a real closed field containing \mathbb{R} , of transcendence degree 1 over \mathbb{R} . Let S_0 be a real closed subfield of S which realizes every cut of \mathbb{Q} . Is S_0 isomorphic to S ?

More general, let S be the pseudo completion of a real closed field and let $\varphi : S \rightarrow S$ be an R -algebra homomorphism. Let S_0 be a real closed field with $\varphi(S) \subseteq S_0 \subseteq S$. Is S_0 isomorphic to S over R ? In the example above, R is the real closure of $\mathbb{Q}(\mu)$, where μ is infinitesimal and S is the pseudo completion of R with respect to the valuation ring $V :=$ the convex hull of \mathbb{Q} in R . Then $S = \mathbb{R}(\mu)$ also is the completion in stages of R with respect to $\{V\}$.

REFERENCES

- [vdD1] L. van den Dries; *Tame Topology and O-minimal Structures*; London Mathematical Society Lecture Note Series, 248
- [vdD3] L. van den Dries; *T-convexity and tame extensions II*; J. Symbolic Logic 62 (1997), no. 1, 14-34
- [vdD-Lew] L. van den Dries, A.H. Lewenberg; *T-convexity and tame extensions*; Journal of Symbolic Logic, vol. 60, no. 1, 74-101 (1995)
- [vdD-S] L. van den Dries, P. Speissegger; *The field of reals with multisummable series and the exponential function*; Proc. London Math. Soc. (3) 81 (2000), no. 3, 513-565.
- [Ho] W. Hodges; *Model Theory*; encyclopedia of mathematics and its applications, vol. 42, 1993
- [Ma-St] D. Marker, C. Steinhorn; *Definable types in o-minimal theories*; J. Symb. Logic, vol. 59, 185-198 (1994)
- [PC] S. Prieß-Crampe; *Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen*; Ergebnisse der Mathematik und ihrer Grenzgebiete 98, Springer 1983
- [Ri] P. Ribenboim; *Théorie des valuations*; Les Presses de l'Université de Montréal, 1964
- [vdW] B. L. van der Waerden; *Algebra I*; Springer, 1966

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