Chains of prime ideals in flat algebras over Prüfer domains

An addendum to an article of M. Nagata

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Introduction

A commutative ring A is called **catenary** if for every pair of primes $q \subset q'$ in $\operatorname{Spec} A$ all maximal chains of primes

$$q = q_0 \subset q_1 \subset \ldots \subset q_\ell = q' \quad (1)$$

possess the same finite length $\ell \in \mathbb{N}$.

A catenary ring A is **locally of finite dimension**, i.e., $\operatorname{height} q \neq \infty$ for every $q \in \operatorname{Spec} A$.

The ring A is called **universally catenary** if every finitely generated A-algebra is catenary. Equivalently, A is universally catenary if every polynomial ring $A[X_1, \ldots, X_n]$, $n \in \mathbb{N}$, is catenary.

Let $A|R$ be a finitely generated extension of domains. Such an extension is said to satisfy the **altitude formula** if for every $q \in \operatorname{Spec} A$ lying over some $p \in \operatorname{Spec} R$ with $\operatorname{height} p \neq \infty$ the relation

$$\operatorname{height} q + \operatorname{trdeg}(A/q|R/p) = \operatorname{height} p + \operatorname{trdeg}(A|R) \quad (2)$$

holds. Here for an extension $C|B$ of domains $\operatorname{trdeg}(C|B)$ denotes the transcendence degree of the field extension $\operatorname{Frac} C|\operatorname{Frac} B$.

If R is locally of finite dimension and $A|R$ satisfies the altitude formula, then A is catenary.

The ring R is said to satisfy the altitude formula, if for every prime $p \in \operatorname{Spec} R$ and every domain B finitely generated over R/p the altitude formula holds in $B|R/p$.

Ratcliff's theorem states that a noetherian, catenary ring R is universally catenary if and only if it satisfies the altitude formula ([Mat], Thm. 15.6). In the non-noetherian case it is known, that a universally catenary ring R satisfies the altitude formula ([BDF], Cor. 4.8).

In [Nag] M. Nagata investigated the class of domains A finitely generated over a valuation domain R of finite Krull dimension. He showed that these algebras satisfy the altitude formula (2) and thus are catenary. Consequently Prüfer domains locally of finite dimension are universally catenary.

Let $A|R$ be a finitely generated extension of domains. The knowledge of the fact that all maximal chains

$$0 = q_0 \subset q_1 \subset \ldots \subset q_\ell = q \quad (3)$$

of primes ascending to $q \in \operatorname{Spec} A$ have a common length ℓ (given for example by (2)) often is not enough. It is desirable to have some overview over the set of all maximal prime sequences ascending to q. More precisely, a first step in that direction could be to fix a prime chain

$$0 = p_0 \subset p_1 \subset \ldots \subset p_r = q \cap R$$

in R and to prescribe the numbers $s_i \in \mathbb{N}$ of primes in a chain (3) lying over p_i for each $i = 0, \ldots, r$. The choice of the s_i is restricted at least by the dimensions of the fibres $A \otimes_R kp_i$ and by height q. For a valuation domain R of finite dimension in [Nag] Nagata stated that these are the only restrictions—without giving a proof. In the present note such a proof is provided.

Prime chains

Throughout this section let R be a Prüfer domain with field of fractions K and let A be a finitely generated, flat R-algebra. Note that flatness of A is equivalent to being torsion-free as an R-module, therefore flatness is always present if A is a domain.

The minimal primes $q_0 \in \operatorname{MinSpec} A$ of A lie over $0 \lhd R$: Due to the Going-Down-Theorem in the flat extension $A|R$ every prime $q \in \operatorname{Spec} A$ contains a prime lying over 0.

In the present notes we focus on the case of an equidimensional generic fibre $\operatorname{Spec}(A \otimes_R K)$, i.e., we assume that the dimensions of the factor rings $A \otimes_R K / q_0 \otimes_R K$, $q_0 \in \operatorname{MinSpec} A$, are all equal. This property is inherited by the other fibres of $\operatorname{Spec} A | \operatorname{Spec} R$:

Theorem 1. *Let A be a flat, finitely generated algebra over the Prüfer domain R. If the generic fibre* $\operatorname{Spec}(A \otimes_R K)$ *is equidimensional of dimension $n \in \mathbb{N}$, then all non-empty fibres* $\operatorname{Spec}(A \otimes_R kp)$*, $p \in \operatorname{Spec} R$, are equidimensional of dimension n.*

Proof. Choose any $p \in \operatorname{Spec} R \setminus 0$ such that the fibre $\operatorname{Spec}(A \otimes_R kp)$ is non-empty and consider the flat, finitely generated R_p-algebra $B := A_p$. Choose a minimal prime ideal $\overline{q}$ of B/pB and its foreimage $q \lhd \operatorname{Spec} B$, which is minimal among the primes lying over p. The prime q contains a minimal prime ideal q_0 of B and we have $q_0 \cap R_p = 0$. The integral domain B/q_0 is a finitely generated R_p-algebra. We apply [Nag], Lemma 2.1 and obtain:

$$\operatorname{trdeg}(B/q_0|R_p) = \operatorname{trdeg}(B/q|kp).$$

By assumption $\operatorname{trdeg}(B/q_0|R_p) = \dim(A \otimes_R K)$, thus since $\operatorname{trdeg}(B/q|kp) = \dim B/q$, and since $\overline{q}$ can be choosen arbitrarily, the assertion is verified. □

For a prime $q \in \operatorname{Spec} A$ lying over $p \in \operatorname{Spec} R$ we introduce the quantity

$$\boxed{s(q) := \dim(A \otimes_R K) - \dim(A/q \otimes_{R/p} kp)}\ . \qquad (4)$$

Since all appearing algebras are finitely generated, in the case of a domain A we can rewrite this formula as:

$$s(q) := \operatorname{trdeg}(A|R) - \operatorname{trdeg}(A/q|R/p). \tag{5}$$

The prime q contains a prime q_p minimal among the primes lying over p and q_p in turn contains some $q_0 \in \operatorname{MinSpec} A$. According to the equidimensionality of $A \otimes_R K$ and to [Nag], Lemma 2.1 we have:

$$\dim(A \otimes_R K) = \operatorname{trdeg}(A/q_0|R) = \operatorname{trdeg}(A/q_p|R/p) = \dim(A/q_p \otimes_{R/p} kp).$$

Comparing this formula with (4) we get:

$$s(q) = \operatorname{height}_{A/q_p \otimes_{R/p} kp}(q/q_p). \tag{6}$$

Note that the right-hand side of (6) does not depend on the choice of q_p since the fibre $A \otimes_R kp$ is equidimensional.

We can now formulate the main result of the present note:

Theorem 2. *Let A be a flat, finitely generated algebra over the Prüfer domain R and assume that the generic fibre $A \otimes_R K$ is equidimensional.*

Let $q \in \operatorname{Spec} A$ be a prime such that $\operatorname{height} p \neq \infty$ for $p = q \cap R$. Let $p_0 \subset p_1 \subset \ldots \subset p_\ell = p$ be primes of R and $s_0, \ldots, s_\ell \in \mathbb{N}$. Then the following statements are equivalent:

1. *There exists a chain of primes $q_0 \subset \ldots \subset q_m = q$ in A such that for each $i = 0, \ldots, \ell$ exactly s_i of the primes q_j lie over p_i.*

2. $\sum_{i=0}^{\ell} (s_i - 1) \leq s(q)$.

Proof. 1. $\Rightarrow$ 2. : The implication is verified by induction with respect to ℓ. The case $\ell = 0$ follows directly from (6). In the case $\ell > 0$ the induction hypothesis implies

$$\sum_{i=0}^{\ell-1} (s_i - 1) \leq s(q_k) \tag{7}$$

where $q_k \cap R = p_{\ell-1}$ and $q_{k+1} \cap R = p$. We choose a prime $q' \in \operatorname{Spec} A$ such that $q_k \subset q' \subseteq q_{k+1}$ holds and q'/q_k is minimal among the primes of A/q_k lying over $p/p_{\ell-1}$. An application of [Nag], Lemma 2.1 yields $\operatorname{trdeg}(A/q_k|R/p_{\ell-1}) = \operatorname{trdeg}(A/q'|R/p)$, thus $s(q_k) = s(q')$. Using (7) and (6) we finally obtain

$$\sum_{i=0}^{\ell} (s_i - 1) \leq s(q_k) + (s_\ell - 1) = s(q') + (s_\ell - 1) \leq s(q)$$

as desired.

2. $\Rightarrow$ 1. : Again we perform induction with respect to ℓ, the case $\ell = 0$ being a direct consequence of formula (6).

We prove the assertion for the case $\ell = \operatorname{height} p_\ell$ refining the original chain $p_0 \subset p_1 \subset \ldots \subset p_\ell = p$ if necessary and assigning the multiplicity $s = 1$ to each prime $p' \in \operatorname{Spec} R$ added due to this refinement. The assertion for the non-refined chain follows from that for the refined chain.

Assuming $\ell > 0$ we first choose a maximal chain $q_1 \subset \ldots \subset q_{s_\ell} = q$ of length s_ℓ in A lying over p_ℓ. Next we take some $q' \in \operatorname{Spec} A$ lying over $p_{\ell-1}$ such that the chain $q' \subset q_1$ is maximal, which is possible because $A \otimes_R kp_{\ell-1}$ is noetherian. We apply the altitude formula in the R-algebra A/q_0 for some $q_0 \in \operatorname{MinSpec} A$ with $q_0 \subseteq q'$ and obtain:

$$\begin{aligned} \operatorname{height} q' &= \operatorname{trdeg}(A/q_0|R) + \operatorname{height} p - 1 - \operatorname{trdeg}(A/q'|R/p_{\ell-1}) && (8) \\ \operatorname{height} q &= \operatorname{trdeg}(A/q_0|R) + \operatorname{height} p - \operatorname{trdeg}(A/q|R/p_\ell). && (9) \end{aligned}$$

By the choice of q' we have $\operatorname{height} q = \operatorname{height} q' + 1$ and thus from (8) and (9) we get that $s(q') = s(q_1)$. Since by formula (6) we have $s(q_1) = s(q) - (s_\ell - 1)$ we obtain

$$\sum_{i=0}^{\ell-1}(s_i - 1) \leq s(q) - (s_\ell - 1) = s(q')$$

and the induction hypothesis yields the existence of a chain $q_0 \subset \ldots \subset q_m = q'$ such that exactly s_i of the q_j lie over p_i for every $i = 0, \ldots, (\ell - 1)$. □

Theorem 2 yields the following property of flat R-schemes $\mathcal{X}$, that is useful when specializing cycles of the generic fibre $\mathcal{X} \times_R K$ to cycles of some fibre $\mathcal{X} \times_R kp$.

Theorem 3. *Let $\mathcal{X}$ be a flat scheme of finite type over the locally finite dimensional Prüfer domain R, possessing an equidimensional generic fibre. Then for every point $y \in \mathcal{X} \times_R kp$ there exists some point $x \in \mathcal{X} \times_R K$ such that y lies in the Zariski closure of $\{x\}$ in $\mathcal{X}$ and $\dim \mathcal{O}_{\mathcal{X}\times kp,y} = \dim \mathcal{O}_{\mathcal{X}\times K,x}$ holds.*

Proof. Choose some open affine neighborhood $U = \operatorname{Spec} A$ of y. For every $p \in \operatorname{Spec} R \backslash 0$ with $p \subseteq y \cap R$ let $s_p := 1$. For $p = 0$ let $s_0 := \dim \mathcal{O}_{\mathcal{X}\times kp,y} = s(y)$. Theorem 2 yields the existence of a chain of primes $q_0 \subset \ldots \subset q_\ell = y$ in A such that the s_0 primes $q_0, \ldots, q_{s_0-1}$ lie over $0 \lhd R$. Since

$$\sum_{p \subseteq y \cap R} (s_p - 1) = s(y)$$

this chain is maximal and we obtain $\operatorname{height} q_{s_0-1} = \dim \mathcal{O}_{\mathcal{X}\times kp,y}$ as desired. □

References

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