TOROIDALIZATION OF GENERATING SEQUENCES IN DIMENSION TWO FUNCTION FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. We give a characteristic free proof of the main result of [5] concerning toroidalization of generating sequences of valuations in dimension two function fields. We show that when an extension of two dimensional algebraic regular local rings $R \subset S$ satisfies the conclusions of the Strong Monomialization theorem of Cutkosky and Piltant, the map between generating sequences in R and S has a toroidal structure.

1. Introduction

The aim of this paper is to prove the main result of [5] in positive characteristic. We start by recalling the set-up and the necessary definitions.

Let $\mathbf{k}$ be an algebraically closed field, and let K be an algebraic function field over $\mathbf{k}$. We say that a subring R of K is *algebraic* if R is essentially of finite type over $\mathbf{k}$. We will denote the maximal ideal of a local ring R by m_R.

Let K^*/K be a finite separable extension of algebraic function fields of transcendence degree 2 over $\mathbf{k}$. Let ν^* be a $\mathbf{k}$-valuation of K^* with valuation ring V^* and value group Γ^*. Let ν be the restriction of ν^* to K with valuation ring V and value group Γ. Consider an extension of algebraic regular local rings $R \subset S$ where R has quotient field K, S has quotient field K^*, R is dominated by S and S is dominated by V^* (i.e., $m_V \cap R = m_R$ and $m_{V^*} \cap S = m_S$).

Let $\Gamma_+ = \nu(R \backslash \{0\})$ be the semigroup of Γ consisting of the values of nonzero elements of R. For $\gamma \in \Gamma_+$, let $I_\gamma = \{f \in R \mid \ \nu(f) \geq \gamma\}$. A (possibly infinite) sequence $\{Q_i\}$ of elements of R is a *generating sequence* of ν [8] if for every $\gamma \in \Gamma_+$ the ideal I_γ is generated by the set

$$\{\prod_i {Q_i}^{a_i} \mid \ a_i \in \mathbb{N}_0, \ \sum_i a_i \nu(Q_i) \geq \gamma\}.$$

A generating sequence of ν is *minimal* if none of its proper subsequences is a generating sequence of ν. A generating sequence of ν^* in S can be defined similarly.

Generating sequences provide a very useful tool in the study of algebraic surfaces (cf. [2, 3, 4, 6, 8, 9] and the literature cited there).

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Let (u, v) be a regular system of parameters (s.o.p.) of R, and let $R' = R\left[\frac{u}{v}\right]_m$, where m is a prime ideal of $R\left[\frac{u}{v}\right]$ such that $m \cap R = m_R$. We say that $R \to R'$ is a *quadratic* transform. If furthermore ν dominates R' we say that $R \to R'$ is a quadratic transform *along* ν.

The main result of [5], which we recall below, gives a nice structure theorem for generating sequences of ν and ν^*, when $\mathbf{k}$ has characteristic zero. We refer to Section 2 of this paper or to Section 2 of [5] for the precise definition of toroidal structure.

Theorem 1.1. [5, 8.1] *Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, and let K^*/K be a finite extension of algebraic function fields of transcendence degree 2 over $\mathbf{k}$. Let ν^* be a $\mathbf{k}$-valuation of K^* with valuation ring V^*, and let ν be the restriction of ν^* to K. Suppose that $R \subset S$ is an extension of algebraic regular local rings with quotient fields K and K^* respectively, such that V^* dominates S and S dominates R. Then there exist sequences of quadratic transforms $R \to \bar{R}$ and $S \to \bar{S}$ along ν^* such that $\bar{S}$ dominates $\bar{R}$ and the map between generating sequences of ν and ν^* in $\bar{R}$ and $\bar{S}$ respectively, has a toroidal structure.*

The goal of this paper is to find a toroidal structure for generating sequences of ν and ν^* when $\mathbf{k}$ has characteristic $\mathbf{p} > 0$.

Cutkosky and Piltant proved that Strong Monomialization holds in positive characteristic, provided that V^*/V is defectless [3, 7.3, 7.35]. The *defect* is an invariant of ramification theory of valuations, and it is a power of $\mathbf{p}$. We refer the reader to Section 7.1 of [3] for the precise definition. We have that V^*/V is defectless whenever Γ^* (and Γ) are finitely generated [3, 7.3]. The only case in which Γ^* is not finitely generated is when it is a non-discrete subgroup of $\mathbb{Q}$. Furthermore, V^*/V is always defectless when $\mathbf{k}$ has characteristic zero. Strong Monomialization may not hold if the extension V^*/V has a defect. See [3, 7.38] for an example. Since in our work we apply Strong Monomialization, we need to assume that V^*/V is defectless.

When Γ^* is a non-discrete subgroup of $\mathbb{Q}$ (which is the essential and subtle case), Strong Monomialization states that there exist sequences of quadratic transforms $R \to R_1$ and $S \to S_1$ along ν^* such that ν^* dominates S_1, S_1 dominates R_1, and there are regular parameters (u, v) in R_1 and (x, y) in S_1, such that the inclusion $R_1 \subset S_1$ is given by

$$\begin{aligned} u &= x^t \delta \\ v &= y, \end{aligned} \tag{1.1}$$

where t is a positive integer and δ is a unit in S_1.

Observe that we can choose $u, v \in R_1$ (resp. $x, y \in S_1$) to be the first two members of a generating sequence of ν (resp. ν^*). Therefore, (1.1) exhibits a toroidal structure of the map between the first two elements of such generating sequences.

The definition of toroidal structures of generating sequences of ν and ν^* is given in Section 2. Our main theorem is stated as follows.

Theorem 1.2 (Theorem 9.1)**.** *Let $\mathbf{k}$ be an algebraically closed field of characteristic $\mathbf{p} > 0$, and let K^*/K be a finite separable extension of algebraic function fields of* *transcendence degree 2 over* $\mathbf{k}$. *Let* ν^* *be a* $\mathbf{k}$*-valuation of* K^* *with valuation ring* V^*, *and let* ν *be the restriction of* ν^* *to* K, *with valuation ring* V. *Assume that* V^*/V *is defectless. Suppose that* $R \subset S$ *is an extension of algebraic regular local rings with quotient fields* K *and* K^* *respectively, such that* V^* *dominates* S *and* S *dominates* R. *Then there exist sequences of quadratic transforms* $R \to \bar{R}$ *and* $S \to \bar{S}$ *along* ν^* *such that* $\bar{S}$ *dominates* $\bar{R}$ *and the map between generating sequences of* ν *and* ν^* *in* $\bar{R}$ *and* $\bar{S}$ *respectively, has a toroidal structure.*

We prove the theorem by analyzing the different types of valuations of K^*. In most cases, the result follows from a standard application of the Strong Monomialization theorem. These cases are analyzed in Section 3. The rest of the paper is devoted to the essential case, when Γ^* is a non-discrete subgroup of $\mathbb{Q}$. We will briefly describe below the main steps of the proof in this case.

We first remark that the methods of [5] can not be extended to positive characteristic. The main obstruction is that the "key lemma" [5, 8.2] no longer holds; that is, the strong monomial form may not be preserved when we apply the quadratic transforms of [5, 8.2]. In this paper we use the sequences of quadratic transforms of the "algorithm" described in Section 7.4 of [3]. This algorithm is recalled in Section 7. The essential point is that the strong monomial form (1.1) is eventually "stable" along the algorithm (see Theorem 7.2).

Other technical difficulties arise from the fact that we used étale extensions in several places in [5], but such extensions are no longer regular rings when we work in positive characteristic. Therefore we do not use étale extensions in this paper.

Let (u, v) be a regular system of parameters in R and let $\{\delta_i\}_{i>0} \subset R$ be a sequence of units such that the residue of δ_i is 1 for all $i > 0$. In Section 4 we construct a sequence of *jumping polynomials* $\{T_i\}_{i\geq 0}$ in R corresponding to (u, v) and to the units $\{\delta_i\}_{i>0}$. By normalizing, we may assume that $\nu(u) = 1$. We let $T_0 = u$ and $T_1 = v$. Write $\nu(v) = p_1/q_1$, where p_1 and q_1 are coprime positive integers. For each $i \geq 1$, we define T_{i+1} recursively. Let p_{i+1} and q_{i+1} be the coprime positive integers defined by

$$\nu(T_{i+1}) = q_i \nu(T_i) + \frac{1}{q_1 \cdots q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

This construction is a generalization of the one given in [5], where the sequence of units was trivial; that is, $\delta_i = 1$ for all $i > 0$. By allowing non trivial units in the jumping polynomials we recover some useful results of [5], avoiding the use of étale extensions. Jumping polynomials corresponding to a trivial sequence of units are very similar to Favre and Jonsson's *key polynomials* [4], whereas the idea of key polynomials is originally due to MacLane [7].

We observe that the above collection of jumping polynomials $\{T_i\}_{i\geq 0}$ forms a generating sequence of ν in R (Theorem 5.4). Furthermore, we can select a subsequence that forms a minimal generating sequence (Theorem 5.7).

Our proof of Theorem 1.2 proceeds as follows. We may assume that R has regular parameters (u, v), and S has regular parameters (x, y) such that the inclusion $R \subset S$

satisfies

$$\begin{aligned} u &= x^t\delta \\ v &= y, \end{aligned} \tag{1.2}$$

where t is a positive integer and δ is a unit in S.

We consider the sequence $\{T_i\}_{i\geq 0}$ of jumping polynomials in R corresponding to (u, v) and to the trivial sequence of units. We consider the sequence $\{T_i'\}_{i\geq 0}$ of jumping polynomials in S corresponding to (x, y) and to the sequence of units given by appropriate powers of δ. Then $\{T_i'\}_{i\geq 0}$ forms a generating sequence of ν^* in S.

Let $Q_k = q_1 \cdots q_k$ for $k > 0$. We show in Theorem 5.9 that if Q_k and t are relatively prime for all $k > 0$, then $T_i = T_i'$ for all $i > 0$. The theorem is proved in this case.

Otherwise we construct appropriate sequences of quadratic transforms $R \to R'$ and $S \to S'$, such that the inclusion $R' \subset S'$ contradicts the stable form of strong monomialization. This step is the main part of our argument, and it requires a very explicit description of the quadratic transforms that we perform. Several crucial preparatory results are discussed in Section 8.

Last we remark that the proof of Theorem 1.2 applies also when $\mathbf{k}$ has characteristic zero, thus providing an alternative argument for Theorem 1.1.

2. Statement of the result

Let $\mathbf{k}$ be an algebraically closed field of characteristic $\mathbf{p} > 0$ and let K^*/K be a finite separable extension of algebraic function fields of transcendence degree 2 over $\mathbf{k}$. Let ν^* be a $\mathbf{k}$-valuation of K^* with valuation ring V^* and value group Γ^* and let ν be the restriction of ν^* to K with valuation ring V and value group Γ.

Suppose that S is an algebraic regular local ring with quotient field K^* which is dominated by V^* and R is an algebraic regular local ring with quotient field K which is dominated by S. We will show that there exist sequences of quadratic transforms $R \to R'$ and $S \to S'$ along ν^* such that S' dominates R' and the map between generating sequences of S' and R' has the following toroidal structure (cf. Section 2 of [5]).

(1) If ν^* is divisorial then $R' = V$ and $S' = V^*$ with regular parameters $u \in R'$ and $x \in S'$ such that $u = x^a\gamma$ for some unit $\gamma \in S'$ and for some positive integer a. We also have that $\{u\}$ is a minimal generating sequence of ν and $\{x\}$ is a minimal generating sequence of ν^*.

(2) If ν^* has rank 2 then there exist regular parameters (u, v) in R' and (x, y) in S' such that $\{u, v\}$ is a minimal generating sequence of ν, $\{x, y\}$ is a minimal generating sequence of ν^*, and

$$u = x^a y^b \delta$$

$$v = y^d \gamma$$

for some units $\delta, \gamma \in S'$, and for some nonnegative integers a, b, d such that $ad \neq 0$.

(3) If ν^* has rank 1 and rational rank 2 then there exist regular parameters (u, v) in R' and (x, y) in S' such that $\{u, v\}$ is a minimal generating sequence of ν, $\{x, y\}$ is a minimal generating sequence of ν^*, and

$$u = x^a y^b \delta$$
$$v = x^c y^d \gamma$$

for some units $\delta, \gamma \in S'$, and for some nonnegative integers a, b, c, d such that $ad - bc \neq 0$.

(4) If Γ and Γ^* are non-discrete subgroups of $\mathbb{Q}$ and V^*/V is defectless, then there exist a generating sequence $\{H_l\}_{l\geq 0}$ of ν in R' and regular parameters (x, y) in S' such that

$$H_0 = x^a \gamma$$
$$H_1 = y$$

for some unit $\gamma \in S'$ and for some positive integer a, and $\{x, \{H_l\}_{l>0}\}$ is a minimal generating sequence of ν^* in S'.

(5) If ν is discrete but not divisorial then there exist regular parameters (u, v) in R' and (x, y) in S' such that Γ is generated by $\nu(u)$, Γ^* is generated by $\nu^*(x)$, and $u = x^a\gamma$ for some unit $\gamma \in S'$ and for some positive integer a. Moreover, R' has a non-minimal generating sequence $\{u, \{T_i\}_{i>0}\}$ such that $\{x, \{T_i\}_{i>0}\}$ is a non-minimal generating sequence in S'.

3. Valuations in 2-dimensional function fields

We will prove our main theorem by analyzing the different types of valuations of K^*. A similar analysis was done in [5] (Section 3), but we outline it below for completeness. We refer to [3] (Section 7.2) for the background needed in this section. We recall that Γ and Γ^* are finitely generated except when they are isomorphic to non-discrete subgroups of $\mathbb{Q}$.

3.1. **One dimensional valuations.** By definition, ν^* is divisorial. In this case ν and ν^* are discrete, and V and V^* are iterated quadratic transforms of R and S respectively (see [1, 4.4]).

Let u be a regular parameter of V and let x be a regular parameter of V^*. Then there is a relation

$$u = x^a \gamma$$

where $\gamma \in V^*$ is a unit and a is a positive integer. Since $\{u\}$ is a mimimal generating sequence for V, and $\{x\}$ is a minimal generating sequence for V^* the theorem is proved.

3.2. **Zero dimensional valuations of rational rank 2.** By [3, 7.3] there exist sequences of quadratic transforms $R \to R'$ and $S \to S'$ along ν^* such that R' has regular parameters (u, v), S' has regular parameters (x, y), and

$$u = x^a y^b \delta$$
$$v = x^c y^d \gamma$$

for some units $\delta, \gamma \in S'$ and for some nonnegative integers a, b, c, d such that $ad - bc \neq 0$. Further, $c = 0$ if ν^* has rank two. We also have that $\{\nu(u), \nu(v)\}$ is a rational basis of $\Gamma \otimes \mathbb{Q}$, and $\{\nu^*(x), \nu^*(y)\}$ is a rational basis of $\Gamma^* \otimes \mathbb{Q}$.

Fix $\gamma \in \Gamma_+ = \nu(R' \backslash \{0\})$ and let $I_\gamma = \{f \in R' \mid \nu(f) \geq \gamma\}$. If $f \in I_\gamma$ we can write $f = \sum_{i \geq 1} a_i u^{b_i} v^{c_i}$, where a_i are units in R', b_i and c_i are nonnegative integers, and the terms have increasing value, since $\nu(u)$ and $\nu(v)$ are rationally independent. It follows that $\nu(f) = b_1\nu(u) + c_1\nu(v)$. For $i \geq 1$ we have $b_i\nu(u) + c_i\nu(v) \geq b_1\nu(u) + c_1\nu(v) = \nu(f) \geq \gamma$. Therefore f belongs to the ideal generated by the set $\{u^{b_i} v^{c_i} \mid b_i, c_i \in \mathbb{N}_0, \ b_i\nu(u) + c_i\nu(v) \geq \gamma\}$. This implies that $\{u, v\}$ is a generating sequence of ν in R'. Furthermore, it is minimal. Similarly $\{x, y\}$ is a minimal generating sequence of ν^* in S', and the theorem is proved.

The rest of the paper will be devoted to studying the remaining cases, that is zero dimensional valuations of rational rank 1.

3.3. **Non-discrete zero dimensional valuations of rational rank 1.** We can normalize Γ^* so that it is an ordered subgroup of $\mathbb{Q}$, whose denominators are not bounded, as Γ^* is not discrete. In Example 3, Section 15, Chapter VI of [10], examples are given of two-dimensional algebraic function fields with value group equal to any given subgroup of the rational numbers. This case is much more subtle.

3.4. **Discrete zero dimensional valuations of rational rank 1.** If ν^* is discrete, then ν is also discrete. This case will be handled in the same way as the case of non-discrete zero dimensional valuations of rational rank 1, but the generating sequences of ν^* and ν will not be minimal.

4. Jumping polynomials

Throughout this section we work under the assumption that the value group of ν is a subgroup of $\mathbb{Q}$ and $\operatorname{trdeg}_{\mathbf{k}}(V/m_V) = 0$. We will first generalize the construction of a sequence of jumping polynomials given in [5].

Suppose that (u, v) is a system of regular parameters in R and $\{\delta_i\}_{i>0} \subset R$ is a sequence of units such that the residue of δ_i is 1 for all $i > 0$. Suppose also that the value group Γ is normalized so that $\nu(u) = 1$. Let

$$\begin{cases} T_0 &= u \\ T_1 &= v. \end{cases}$$

Set $q_0 = \infty$ and choose a pair of coprime positive integers (p_1, q_1) so that $\nu(v) = p_1/q_1$. For $i \geq 1$, T_{i+1} is defined recursively as follows. Let

$$T_{i+1} = T_i^{q_i} - \lambda_i \delta_i \prod_{j=0}^{i-1} T_j^{n_{i,j}},$$

where $n_{i,j} < q_j$ are nonnegative integers such that $q_i \nu(T_i) = \nu(\prod_{j=0}^{i-1} T_j^{n_{i,j}})$ and $\lambda_i \in \mathbf{k} - \{0\}$ is the residue of $T_i^{q_i}(\prod_{j=0}^{i-1} T_j^{n_{i,j}})^{-1}$.

Writing $\delta_i = 1 + w_i$, for some $w_i \in m_R$, we notice that

$$\nu(T_{i+1}) \geq \min\{\nu(T_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} T_j^{n_{i,j}}), \nu(\lambda_i w_i \prod_{j=0}^{i-1} T_j^{n_{i,j}})\} > q_i \nu(T_i).$$

Therefore we can choose positive integers p_{i+1} and q_{i+1} so that $(p_{i+1}, q_{i+1}) = 1$ and

$$\nu(T_{i+1}) = q_i \nu(T_i) + \frac{1}{q_1 \cdots q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

We will say that $\{T_i\}_{i \geq 0}$ is a sequence of jumping polynomials corresponding to the regular parameters (u, v) and the sequence of units $\{\delta_i\}_{i>0}$. The polynomial T_i will be called the i-th *jumping polynomial* and the value $\nu(T_i)$ will be called the i-th j-*value*. We denote the i-th j-value by β_i and we say that β_i is an *independent* j-value if $q_i \neq 1$. In this case we say that T_i is an independent jumping polynomial.

It is shown in [5, 5.10] that the sequence of jumping polynomials corresponding to the trivial sequence of units $\{1\}_{i>0}$ is well defined, that is, the $n_{i,j}$ above are uniquely determined. The same considerations show that the sequence of jumping polynomials $\{T_i\}_{i \geq 0}$ corresponding to any sequence of units $\{\delta_i\}_{i>0} \subset R$, where $(\delta_i - 1) \in m_R$ for all $i > 0$, is well defined. Furthermore, the values of jumping polynomials have the following properties.

Remark 4.1. For $i > 0$ denote $Q_i = q_1 \cdots q_i$ and set $Q_0 = 1$. If $i > 0$, then

1) $\beta_{i+1} = q_i \beta_i + \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}$,
2) $Q_i \beta_j$ is an integer number for all $j \leq i$,
3) $q_{i+1} \beta_{i+1} \geq \beta_{i+1} > q_i \beta_i \geq \beta_i$,
4) $q_i \beta_i = \sum_{j=0}^{i-1} n_{i,j} \beta_j$.

Assume now that $\{\beta_{i_l}\}_{l \geq 0}$ is the subsequence of all independent j-values. Let $\bar{\beta}_l = \beta_{i_l}$ denote the l-th independent j-value, $\bar{q}_l = q_{i_l}$ and $\bar{p}_l = (p_{i_{l-1}+1} + \cdots + p_{i_l - 1})\bar{q}_l + p_{i_l}$ if $l > 0$. Then the values of independent jumping polynomials have the following properties.

Remark 4.2. For $l > 0$ denote $\bar{Q}_l = \bar{q}_1 \cdots \bar{q}_l$ and set $\bar{Q}_0 = 1$. If $l > 0$, then

1) $\bar{\beta}_{l+1} = \bar{q}_l \bar{\beta}_l + \frac{1}{\bar{Q}_l} \cdot \frac{\bar{p}_{l+1}}{\bar{q}_{l+1}}$ and $\bar{\beta}_1 = \frac{\bar{p}_1}{\bar{q}_1}$,

2) $\bar{Q}_l\beta_{i'}$ is an integer number for all $i' < i_{l+1}$. In particular, $\bar{Q}_l\bar{\beta}_j$ is an integer number for all $j \leq l$,
3) $\bar{q}_{l+1}\bar{\beta}_{l+1} > \bar{\beta}_{l+1} > \bar{q}_l\bar{\beta}_l > \bar{\beta}_l$,
4) $(\bar{p}_l, \bar{q}_l) = 1$.

Let us consider the sequence $\{H_l\}_{l\geq 0}$ of all independent jumping polynomials in R. For all $l \geq 0$ we have $H_l = T_{i_l}$, in particular, $H_0 = u$ and $H_1 = v - \sum_{j=1}^{i_1-1} \lambda_j\delta_j u^{\beta_j}$. Since $n_{i,j} < q_j$, that is, $n_{i,j} = 0$ whenever T_j is not an independent jumping polynomial, the recursive formula for H_{l+1} with $l > 0$ is

$$H_{l+1} = H_l^{\bar{q}_l} - \lambda_{i_l}\delta_{i_l}\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}} - \lambda_{i_l+1}\delta_{i_l+1}\prod_{j=0}^{l} H_j^{n_{i_l+1,i_j}} - \lambda_{i_l+2}\,\delta_{i_l+2}\prod_{j=0}^{l} H_j^{n_{i_l+2,i_j}} - \dots$$

$$\dots - \lambda_{i_{l+1}-1}\,\delta_{i_{l+1}-1}\prod_{j=0}^{l} H_j^{n_{i_{l+1}-1,i_j}} = H_l^{\bar{q}_l} - \lambda_{i_l}\delta_{i_l}\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}} - \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'}\delta_{i'}\prod_{j=0}^{l} H_j^{n_{i',i_j}}.$$

Remark 4.3. In general, if R is a 2-dimensional regular local ring dominated by V and (u, v) is a system of regular parameters in R, we may not necessarily have $\nu(u) = 1$. Then in order to define a sequence of jumping polynomials $\{T_i\}_{i\geq 0}$ corresponding to the system of regular parameters (u, v), we introduce the following valuation $\tilde{\nu}$ of K

$$\tilde{\nu}(f) = \frac{\nu(f)}{\nu(u)}$$

for all $f \in K$. Then $\tilde{\nu}(u) = 1$ and we use the construction above with ν replaced by the equivalent valuation $\tilde{\nu}$. This procedure is equivalent to normalizing the value group Γ so that $\nu(u) = 1$.

5. Properties of jumping polynomials

In this section assumptions and notations are as in Section 4. Our first goal is to show that sequences of jumping polynomials form generating sequences of valuations. See [3, 4, 7, 8] for more considerations on this topic.

Let $\Gamma_+ = \nu(R\backslash\{0\})$ be the semigroup of Γ consisting of the values of nonzero elements of R. For $\gamma \in \Gamma_+$, let $I_\gamma = \{f \in R|\ \nu(f) \geq \gamma\}$. Then a possibly infinite sequence $\{Q_i\} \subset R$ is a *generating sequence* of ν if for every $\gamma \in \Gamma_+$ the ideal I_γ is generated by the set $\{\prod_i {Q_i}^{a_i}|\ a_i \in \mathbb{N}_0,\ \sum_i a_i\nu(Q_i) \geq \gamma\}$. A generating sequence of ν is *minimal* if none of its proper subsequences is a generating sequence of ν.

If $\gamma \in \Gamma_+$ we denote by $\mathcal{A}_\gamma$ the ideal of R generated by $\{\prod_{j=0}^{k} T_j^{m_j}|\ k, m_j \in \mathbb{N}_0,$ $\sum_{j=0}^{k} m_j\beta_j \geq \gamma\}$, and we denote by $\mathcal{A}_\gamma^+$ the ideal of R generated by $\{\prod_{j=0}^{k} T_j^{m_j}|\ k, m_j \in \mathbb{N}_0,\ \sum_{j=0}^{k} m_j\beta_j > \gamma\}$. We observe the following basic properties of the ideals $\mathcal{A}_\gamma$ and $\mathcal{A}_\gamma^+$.

1) $\mathcal{A}_\gamma \subset I_\gamma$.

2) If $\gamma_1 < \gamma_2$ then $\mathcal{A}_{\gamma_2} \subset \mathcal{A}^+_{\gamma_1} \subset \mathcal{A}_{\gamma_1}$.
3) $\mathcal{A}_{\gamma_1}\mathcal{A}_{\gamma_2} \subset \mathcal{A}_{\gamma_1+\gamma_2}$ and $\mathcal{A}^+_{\gamma_1}\mathcal{A}_{\gamma_2} \subset \mathcal{A}^+_{\gamma_1+\gamma_2}$.
4) Let $\beta = \min(\beta_0, \beta_1)$. If $f \in m_R$ then $f = uf_1 + vf_2$ for some $f_1, f_2 \in R$ and therefore $f \in \mathcal{A}_\beta$. Thus $m_R \subset \mathcal{A}_\beta \subset \mathcal{A}^+_0$.
5) For any $\gamma \in \Gamma_+$ there exists $\gamma' \in \Gamma_+$ such that $\mathcal{A}^+_\gamma = \mathcal{A}_{\gamma'}$. Indeed, notice that the set $\{\alpha \in \Gamma_+ | \gamma < \alpha \leq \gamma + \beta_0\}$ is finite, since it is bounded from above, and it is nonempty. Then set $\gamma' = \min\{\alpha \in \Gamma_+ | \gamma < \alpha \leq \gamma + \beta_0\}$.

Lemma 5.1. *Let $f = \prod_{j=0}^k T_j^{m_j}$, where $k, m_0, \dots, m_k \in \mathbb{N}_0$, and let $\gamma = \nu(f) = \sum_{j=0}^k m_j\beta_j$. There exist nonnegative integers $d_0, d_1, \dots, d_k$ such that $\sum_{j=0}^k d_j\beta_j = \gamma$ and $d_j < q_j$ for all $0 \leq j \leq k$, a unit $\mu \in R$ and $f' \in \mathcal{A}^+_\gamma$ such that $f = \mu\prod_{j=0}^k T_j^{d_j} + f'$.*

Proof. We apply induction on k. If $k = 0$ then $f = T_0^{m_0}$ is the required presentation. If $k > 0$, write $m_k = rq_k + d_k$ for some $r \geq 0$ and $0 \leq d_k < q_k$. Recall that $\nu(\prod_{j=0}^{k-1} T_j^{n_{k,j}}) = q_k\beta_k$ and $\nu(T_{k+1}) > q_k\beta_k$. Thus

$$T_k^{rq_k} = (T_{k+1} + \lambda_k\delta_k \prod_{j=0}^{k-1} T_j^{n_{k,j}})^r = \lambda_k^r\delta_k^r \prod_{j=0}^{k-1} T_j^{rn_{k,j}} + h,$$

where $h \in \mathcal{A}^+_{rq_k\beta_k}$. Furthermore, since $d_k\beta_k + \sum_{j=0}^{k-1} m_j\beta_j = \gamma - rq_k\beta_k$ we have that $hT_k^{d_k}\prod_{j=0}^{k-1} T_j^{m_j} \in \mathcal{A}^+_{rq_k\beta_k}\mathcal{A}_{\gamma - rq_k\beta_k} \subset \mathcal{A}^+_\gamma$.

Let $g = \prod_{j=0}^{k-1} T_j^{m_j + rn_{k,j}}$ and let $\alpha = \gamma - d_k\beta_k$. Notice that $\nu(g) = \alpha$. Then by the inductive assumption there exist nonnegative integers $d_0, d_1, \dots, d_{k-1}$ such that $\sum_{j=0}^{k-1} d_j\beta_j = \alpha$ and $d_j < q_j$ for all $0 \leq j \leq k-1$, a unit $\mu' \in R$ and $g' \in \mathcal{A}^+_\alpha$ such that $g = \mu'\prod_{j=0}^{k-1} T_j^{d_j} + g'$. We also notice that $g'T_k^{d_k} \in \mathcal{A}^+_\alpha\mathcal{A}_{d_k\beta_k} \subset \mathcal{A}^+_\gamma$. Thus

$$f = T_k^{d_k}(\lambda_k^r\delta_k^r \prod_{j=0}^{k-1} T_j^{rn_{k,j}} + h) \prod_{j=0}^{k-1} T_j^{m_j} = T_k^{d_k}(\lambda_k^r\delta_k^r g + h\prod_{j=0}^{k-1} T_j^{m_j}) =$$
$$= \lambda_k^r\delta_k^r\mu' \prod_{j=0}^{k} T_j^{d_j} + \lambda_k^r\delta_k^r g' T_k^{d_k} + hT_k^{d_k}\prod_{j=0}^{k-1} T_j^{m_j} = \mu\prod_{j=0}^{k} T_j^{d_j} + f',$$

where $\mu = \lambda_k^r\delta_k^r\mu'$ is a unit in R and $f' \in \mathcal{A}^+_\gamma$. □

Remark 5.2. The integers $d_0, d_1, \dots, d_k$ of Lemma 5.1 depend only on γ: there exists a unique $(k+1)$-tuple of nonnegative integers $d_0, d_1, \dots, d_k$ such that $\sum_{j=0}^k d_j\beta_j = \gamma$ and $d_j < q_j$ for all $0 \leq j \leq k$.

The statement above is equivalent to the one claiming that if $\sum_{j=0}^k c_j\beta_j = 0$ for some integer coefficients $-q_j < c_j < q_j$ then $c_j = 0$ for all $0 \leq j \leq k$. We refer the reader to Proposition 5.8 of [5] for the proof.

Lemma 5.3. *If $\gamma \in \Gamma_+$, then $I_\gamma = \mathcal{A}_\gamma$.*

Proof. We only need to check that $I_\gamma \subset \mathcal{A}_\gamma$ for all $\gamma \in \Gamma_+$.

Let $\gamma \in \Gamma_+$ and let $f \in I_\gamma$. We will show that $f \in \mathcal{A}_\gamma$. First notice that if $f \in \mathcal{A}_\alpha$ for some $\alpha \in \Gamma_+$ then $\alpha \leq \nu(f)$. Thus the set $\Omega = \{\alpha \in \Gamma_+ | \, f \in \mathcal{A}_\alpha\}$ is finite since it is bounded from above and it is nonempty since $f \in \mathcal{A}_0$. We choose σ to be the maximal element of Ω. Then there exists a presentation

$$f = \sum_{l=1}^{N} g_l \prod_{j=0}^{k} T_j^{m_{l,j}} + f',$$

where $\sum_{j=0}^k m_{l,j}\beta_j = \sigma$ for all $1 \leq l \leq N$, $g_l \in R$ for all $1 \leq l \leq N$, and $f' \in \mathcal{A}_\sigma^+$.

We now apply Lemma 5.1 to $\prod_{j=0}^k T_j^{m_{l,j}}$ for all $1 \leq l \leq N$. We get $\prod_{j=0}^k T_j^{m_{l,j}} = \mu_l \prod_{j=0}^k T_j^{d_j} + h_l$, where $\mu_l \in R$ is a unit, $\sum_{j=0}^k d_j\beta_j = \sigma$ and $h_l \in \mathcal{A}_\sigma^+$. Thus

$$f = (\sum_{l=1}^{N} \mu_l g_l) \prod_{j=0}^{k} T_j^{d_j} + \sum_{l=1}^{N} h_l g_l + f' = \mu \prod_{j=0}^{k} T_j^{d_j} + h,$$

where $h \in \mathcal{A}_\sigma^+$. If $\mu \in m_R$ then $\mu \prod_{j=0}^k T_j^{d_j} \in \mathcal{A}_0^+ \mathcal{A}_\sigma \subset \mathcal{A}_\sigma^+$ and therefore $f \in \mathcal{A}_\sigma^+$. Let $\alpha \in \Gamma_+$ be such that $\mathcal{A}_\sigma^+ = \mathcal{A}_\alpha$. Then $\alpha > \sigma$ and $f \in \mathcal{A}_\alpha$, a contradiction to the choice of σ. So μ is a unit in R and $\nu(\mu) = 0$. Then $\nu(f) = \min(\nu(\mu \prod_{j=0}^k T_j^{d_j}), \nu(h)) = \sigma$. Thus we get that $\sigma \geq \gamma$, and so $f \in \mathcal{A}_\sigma \subset \mathcal{A}_\gamma$. □

Theorem 5.4. *$\{T_i\}_{i \geq 0}$ is a generating sequence in R.*

Proof. The statement follows at once from Lemma 5.3 and the definition of generating sequences. □

5.1. **Non-discrete case.** We will now assume that the value group of ν is not finitely generated and, therefore, the sequence of independent jumping polynomials $\{H_l\}_{l \geq 0}$ is infinite. If $\gamma \in \Gamma_+$ denote by $\mathcal{B}_\gamma$ the ideal of R generated by $\{\prod_{j=0}^k H_j^{m_j} | \, k, m_j \in \mathbb{N}_0, \, \sum_{j=0}^k m_j \bar{\beta}_j \geq \gamma\}$ and denote by $\mathcal{B}'_\gamma$ the ideal of R generated by $\{\prod_{j=1}^k H_j^{m_j} | \, k, m_j \in \mathbb{N}_0, \, \sum_{j=1}^k m_j \bar{\beta}_j \geq \gamma\}$. We notice that $\mathcal{B}_{\gamma_1}\mathcal{B}_{\gamma_2} \subset \mathcal{B}_{\gamma_1+\gamma_2}$, $\mathcal{B}'_{\gamma_1}\mathcal{B}'_{\gamma_2} \subset \mathcal{B}'_{\gamma_1+\gamma_2}$, and $\mathcal{B}'_\gamma \subset \mathcal{B}_\gamma \subset \mathcal{A}_\gamma$ for all $\gamma, \gamma_1, \gamma_2 \in \Gamma_+$.

Lemma 5.5. *Suppose that Γ is a non-discrete subgroup of $\mathbb{Q}$. Then $T_k \in \mathcal{B}_{\beta_k}$ for all $k \geq 0$. Furthermore, if $\bar{p}_1 = 1$ then $H_0 \in \mathcal{B}'_{\bar{\beta}_0}$.*

Proof. We fix $k \geq 0$. Since $\{H_l\}_{l \geq 0}$ is infinite there exists l such that $i_{l-1} < k \leq i_l$. Then since $q_j = 1$ for all $k \leq j < i_l$ we have

$$T_{i_l} = T_k - \lambda_k \delta_k \prod_{j=0}^{k-1} T_j^{n_{k,j}} - \lambda_{k+1}\delta_{k+1} \prod_{j=0}^{k} T_j^{n_{k+1,j}} - \cdots - \lambda_{i_l-1}\delta_{i_l-1} \prod_{j=0}^{i_l-2} T_j^{n_{i_l-1,j}}.$$

We notice that $n_{i,j} = 0$ whenever T_j is not an independent jumping polynomial. Thus

$$T_k = T_{i_l} + \sum_{i'=k}^{i_l-1} \lambda_{i'} \delta_{i'} \prod_{j=0}^{i'-1} T_j^{n_{i',j}} = H_l + \sum_{i'=k}^{i_l-1} \lambda_{i'} \delta_{i'} \prod_{j=0}^{l-1} H_j^{n_{i',i_j}},$$

where $\bar{\beta}_l \geq \beta_k$ and $\sum_{j=0}^{l-1} n_{i',i_j} \bar{\beta}_j = \nu(\prod_{j=0}^{l-1} H_j^{n_{i',i_j}}) = \nu(\prod_{j=0}^{i'-1} T_j^{n_{i',j}}) = q_{i'}\beta_{i'} = \beta_{i'} \geq \beta_k$ for all $k \leq i' < i_l$. So $T_k \in \mathcal{B}_{\beta_k}$.

Assume now that $\bar{p}_1 = 1$. Then since $\bar{p}_1 = (p_1 + \cdots + p_{i_1-1})\bar{q}_1 + p_{i_1}$ we have $i_1 = 1$, $1 = \bar{p}_1 = p_1$, $\bar{q}_1 = q_1$ and $H_0 = u$, $H_1 = v$. Also

$$H_2 = v^{q_1} - \lambda_1 \delta_1 u - \sum_{i'=2}^{i_2-1} \lambda_{i'} \delta_{i'} u^{n_{i',0}} v^{n_{i',1}},$$

where $n_{i',1} < q_1$ for all $2 \leq i' \leq i_2 - 1$. Since $\nu(u^{n_{i',0}} v^{n_{i',1}}) = \beta_{i'} > q_1\beta_1 = 1$ and $\nu(v^{n_{i',1}}) = n_{i',1}/q_1 < 1$ we see that $n_{i',0} > 0$ for all $2 \leq i' \leq i_2 - 1$. Thus

$$H_0 = u = (v^{q_1} - H_2)(\lambda_1 \delta_1 + \sum_{i'=2}^{i_2-1} \lambda_{i'} \delta_{i'} u^{n_{i',0}-1} v^{n_{i',1}})^{-1} = (H_1^{q_1} - H_2)\Delta,$$

where Δ is a unit in R and $q_1\beta_1 = 1$, $\beta_2 > 1$. So $H_0 \in \mathcal{B}'_1 = \mathcal{B}'_{\bar{\beta}_0}$. ☐

Lemma 5.6. *Suppose that* Γ *is a non-discrete subgroup of* $\mathbb{Q}$. *If* $\gamma \in \Gamma_+$, *then* $\mathcal{B}_\gamma = \mathcal{A}_\gamma$. *Furthermore, if* $\bar{p}_1 = 1$ *then* $\mathcal{B}'_\gamma = \mathcal{A}_\gamma$.

Proof. To prove that $\mathcal{B}_\gamma = \mathcal{A}_\gamma$ it suffices to show that if $\sum_{j=0}^k m_j \beta_j \geq \gamma$ then $\prod_{j=0}^k T_j^{m_j} \in \mathcal{B}_\gamma$. By Lemma 5.5 we have that $\prod_{j=0}^k T_j^{m_j} \in \prod_{j=0}^k \mathcal{B}_{\beta_j}^{m_j} \subset \mathcal{B}_\gamma$.

Now assume that $\bar{p}_1 = 1$ and consider $\prod_{j=0}^k H_j^{m_j}$ with $\sum_{j=0}^k m_j \bar{\beta}_j \geq \gamma$. By Lemma 5.5 we have that $\prod_{j=0}^k H_j^{m_j} \in \prod_{j=0}^k (\mathcal{B}'_{\bar{\beta}_j})^{m_j} \subset \mathcal{B}'_\gamma$. Thus $\mathcal{B}'_\gamma = \mathcal{B}_\gamma = \mathcal{A}_\gamma$. ☐

Theorem 5.7. *If* Γ *is a non-discrete subgroup of* $\mathbb{Q}$ *then* $\{H_l\}_{l\geq 0}$ *forms a generating sequence in* R. *Moreover, if* $\bar{p}_1 \neq 1$ *then* $\{H_l\}_{l\geq 0}$ *is a minimal generating sequence in* R *and if* $\bar{p}_1 = 1$ *then* $\{H_l\}_{l>0}$ *forms a minimal generating sequence in* R.

Proof. It follows from Lemmas 5.3 and 5.6 that $I_\gamma = \mathcal{B}_\gamma$ for all $\gamma \in \Gamma_+$. Thus $\{H_l\}_{l\geq 0}$ is a generating sequence for ν. If $\bar{p}_1 = 1$ then we have $I_\gamma = \mathcal{B}'_\gamma$ for all $\gamma \in \Gamma_+$. Thus in this case $\{H_l\}_{l>0}$ forms a generating sequence for ν.

To prove the statement about minimality we introduce the following notation: for $k \geq 0$ denote by $\bar{\Gamma}_k$ the group generated by $\{\bar{\beta}_j\}_{j=0}^k$, denote by Φ_k the semigroup generated by $\{\bar{\beta}_j\}_{j=0}^k$, and denote by $\Phi_{\hat{k}}$ the semigroup generated by $\{\bar{\beta}_j\}_{j=0}^{k-1} \cup \{\bar{\beta}_j\}_{j>k}$. We will prove first that if $k > 0$, then $\bar{\beta}_k \notin \Phi_{\hat{k}}$. Therefore, $\Gamma_+ \neq \Phi_{\hat{k}}$ for all $k > 0$.

It is shown in [5, 5.6] that $\bar{\Gamma}_k = (1/\bar{Q}_k)\mathbb{Z}$ for all $k \geq 0$. Thus $\bar{\Gamma}_{k-1} \neq \bar{\Gamma}_k$ and we have $\bar{\beta}_k \notin \bar{\Gamma}_{k-1}$. So $\bar{\beta}_k \notin \Phi_{k-1}$. On the other hand $\bar{\beta}_{k+j} > \bar{\beta}_k$ for all $j, k > 0$, so if $\bar{\beta}_k \in \Phi_{\hat{k}}$ for some $k > 0$ then $\bar{\beta}_k \in \Phi_{k-1}$. Thus $\bar{\beta}_k \notin \Phi_{\hat{k}}$ for all $k > 0$.

It follows that if some subsequence $\mathcal{H} = \{H_{l_j}\}_{j\geq 0}$ is a generating sequence for ν then $\{H_l\}_{l>0} \subset \mathcal{H}$, since $\{\nu(H_{l_j})\}_{j\geq 0}$ needs to generate Γ_+.

Assume now that $\bar{p}_1 \neq 1$. Then $\bar{\beta}_0$ is not a multiple of $\bar{\beta}_1$ and for all $j \geq 2$ we have $\bar{\beta}_j \geq \bar{\beta}_2 > \bar{q}_1\bar{\beta}_1 = \bar{p}_1 > 1 = \bar{\beta}_0$. Thus $\bar{\beta}_0 \notin \Phi_{\hat{0}}$ and, therefore, any generating sequence $\{H_{l_j}\}_{j\geq 0}$ has to contain H_0. □

5.2. **Discrete case.** Suppose now that the value group of ν is isomorphic to $\mathbb{Z}$. Then by [8] (p. 154) every generating sequence of ν is infinite and there are no minimal generating sequences in R. In our construction we will be mostly concerned with the situation when R has a system of regular parameters (u, v) such that $\nu(u)$ generates Γ. In this case sequences of jumping polynomials in R have the following property.

Theorem 5.8. *Suppose that $\Gamma \cong \mathbb{Z}$ and that β_0 generates Γ. Then any infinite subsequence $\{T_{i_j}\}_{j\geq 0}$ containing T_0 is a generating sequence in R.*

Proof. We will follow the same line of arguments as in Lemmas 5.5 and 5.6. If $\gamma \in \Gamma_+$ denote by $\mathcal{A}'_\gamma$ the ideal of R generated by $\{\prod_{j=0}^k T_{i_j}^{m_j} \mid k, m_j \in \mathbb{N}_0, \sum_{j=0}^k m_j\beta_{i_j} \geq \gamma\}$. We will show that $\mathcal{A}'_\gamma = \mathcal{A}_\gamma$ for all $\gamma \in \Gamma_+$. Then Lemma 5.3 implies that $\{T_{i_j}\}_{j\geq 0}$ is a generating sequence of ν.

We notice first that since Γ is generated by $\beta_0 = 1$, for all $i > 0$ we have $\beta_i \in \mathbb{N}$ and $q_i = 1$. It follows that $n_{i,0} = \beta_i$ and $n_{i,i'} = 0$ for all $0 < i' < i$.

Fix $k \geq 0$. Since $\{T_{i_j}\}_{j\geq 0}$ is infinite there exists j such that $i_{j-1} < k \leq i_j$. Then we have

$$T_k = T_{i_j} + \lambda_{i_j-1}\delta_{i_j-1}T_0^{\beta_{i_j-1}} + \cdots + \lambda_k\delta_k T_0^{\beta_k},$$

where $\nu(T_{i_j}) = \beta_{i_j} \geq \beta_k$ and $\nu(T_0^{\beta_{i'}}) = \beta_{i'} \geq \beta_k$ for all $k \leq i' \leq i_j - 1$. Thus $T_k \in \mathcal{A}'_{\beta_k}$. To prove that $\mathcal{A}'_\gamma = \mathcal{A}_\gamma$ it suffices to show that if $\sum_{j=0}^k m_j\beta_j \geq \gamma$ then $\prod_{j=0}^k T_j^{m_j} \in \mathcal{A}'_\gamma$. This is true since $\prod_{j=0}^k T_j^{m_j} \in \prod_{j=0}^k (\mathcal{A}'_{\beta_j})^{m_j} \subset \mathcal{A}'_\gamma$.

□

Our next goal is to understand the relationship between jumping polynomials in R and S when the inclusion $R \subset S$ satisfies the conclusions of the Strong Monomialization theorem. We fix regular parameters (u, v) in R. Let $\{T_i\}_{i\geq 0}$ be the sequence of jumping polynomials in R corresponding to (u, v) and the trivial sequence of units. For all $i > 0$ let (p_i, q_i) be the pair of coprime integers defined in the construction of the jumping polynomials, let λ_i be the scalar and for $0 \leq j < i$ let $n_{i,j}$ be the powers defined in the construction of $\{T_i\}_{i\geq 0}$. We also fix regular parameters (x, y) in S and a unit $\delta \in S$ such that $(\delta - 1) \in m_S$. Let $\{T'_i\}_{i\geq 0}$ be the sequence of jumping polynomials in S corresponding to (x, y) and the sequence of units $\{\delta^{n_{i,0}}\}_{i>0}$. For all $i > 0$ let (p'_i, q'_i) be the pair of coprime integers defined in the construction of the jumping polynomials, let λ'_i be the scalar and for $0 \leq j < i$ let $n'_{i,j}$ be the powers defined in the construction of $\{T'_i\}_{i\geq 0}$.

Theorem 5.9. *With notations as above, suppose that the inclusion of* 2*-dimensional regular local rings* $R \subset S$ *satisfies the equation*

$$u = x^t \delta$$
$$v = y,$$

where t *is a positive integer. If* $(t, Q_k) = 1$ *for some* $k > 0$ *then*

1) $T'_i = T_i$ *for all* $0 < i \leq k+1$*;*
2) $q'_i = q_i$ *and* $p'_i = tp_i$ *for all* $0 < i \leq k$*;*
3) $n'_{i,j} = n_{i,j}$ *for all* $0 < j < i \leq k$ *and* $n'_{i,0} = tn_{i,0}$ *for all* $0 < i \leq k$*.*

Proof. We may assume that $\nu(u) = \nu^*(u) = 1$. Then $\nu^*(x) = 1/t$ and in order to construct a sequence of jumping polynomials corresponding to the regular parameters (x, y) in S we define the following valuation $\tilde{\nu}$ of K^*:

$$\tilde{\nu}(f) = t\nu^*(f) \text{ for all } f \in K^*.$$

We have $T_0 = u$, $T'_0 = x$, $T_1 = v$ and $T'_1 = y$. Thus $T_0 = \delta T'^{t}_0$ and $T_1 = T'_1$.

Suppose that $(t, q_1) = 1$. The coprime integers p'_1 and q'_1 are such that $p'_1/q'_1 = \tilde{\nu}(y) = t\nu^*(v) = tp_1/q_1$. Since $(tp_1, q_1) = 1$ we get $p'_1 = tp_1$ and $q'_1 = q_1$. Since $n_{1,0} = p_1$ and $n'_{1,0} = p'_1$ we also have $n'_{1,0} = tn_{1,0}$. Furthermore, $T'^{q'_1}_1 = T^{q_1}_1$ and $\delta^{n_{1,0}} T'^{n'_{1,0}}_0 = T^{n_{1,0}}_0$. Then, since the residue of δ is 1 and $T'^{q'_1}_1 T'^{-n'_{1,0}}_0 = T^{q_1}_1 T^{-n_{1,0}}_0 \delta^{n_{1,0}}$, taking the residue of both sides of the above equality we get that $\lambda'_1 = \lambda_1$. Thus $T'_2 = T_2$ and the statement is proved for $k = 1$.

We apply induction on k. Suppose that $(t, Q_k) = 1$. Then $(t, Q_{k-1}) = 1$ and by the inductive assumption we have that $q'_{k-1} = q_{k-1}$, $Q'_{k-1} = Q_{k-1}$ and $T'_j = T_j$ for all $0 < j \leq k$. The coprime integers p'_k and q'_k satisfy the following equality

$$p'_k/q'_k = Q'_{k-1}(\tilde{\nu}(T'_k) - q'_{k-1}\tilde{\nu}(T'_{k-1})) = tQ_{k-1}(\nu^*(T_k) - q_{k-1}\nu^*(T_{k-1})) = tp_k/q_k.$$

Since $(t, q_k) = 1$ we get $p'_k = tp_k$ and $q'_k = q_k$. Since $\tilde{\nu}(T'_0) = 1$, by the construction of jumping polynomials $\{T'_i\}_{i\geq 0}$ in S we get

$$q_k\nu^*(T_k) = t^{-1}q'_k\tilde{\nu}(T'_k) = t^{-1}\sum_{j=0}^{k-1} n'_{k,j}\tilde{\nu}(T'_j) = n'_{k,0}/t + \sum_{j=1}^{k-1} n'_{k,j}\nu^*(T_j).$$

On the other hand by the construction of jumping polynomials $\{T_i\}_{i\geq 0}$ in R we get that $q_k\nu^*(T_k) = \sum_{j=0}^{k-1} n_{k,j}\nu^*(T_j)$ and this representation is unique. Therefore, $n'_{k,0} = tn_{k,0}$ and $n'_{k,j} = n_{k,j}$ for all $0 < j < k$.

Finally, $T'^{q'_k}_k = T^{q_k}_k$ and $\delta^{n_{k,0}} \prod_{j=0}^{k-1} T'^{n'_{k,j}}_j = \prod_{j=0}^{k-1} T^{n_{k,j}}_j$. Then, since the residue of δ is 1 and $T'^{q'_k}_k (\prod_{j=0}^{k-1} T'^{n'_{k,j}}_j)^{-1} = T^{q_k}_k (\prod_{j=0}^{k-1} T^{n_{k,j}}_j)^{-1}\delta^{n_{k,0}}$, taking the residue of both sides of the above equality we get that $\lambda'_k = \lambda_k$. Thus $T'_{k+1} = T_{k+1}$ and the theorem is proved.

□

6. Behavior of jumping polynomials under blow-ups

Throughout this section we work under the assumption that the value group of ν is a subgroup of $\mathbb{Q}$ and $\mathrm{trdeg}_{\mathbf{k}}(V/m_V) = 0$.

We now introduce the notations used in the rest of the paper.

If p and q are positive integers such that $(p, q) = 1$, the Euclidian algorithm for finding the greatest common divisor of p and q can be described as follows:

$$\begin{aligned} r_0 &= f_1 r_1 + r_2 \\ r_1 &= f_2 r_2 + r_3 \\ &\vdots \\ r_{N-2} &= f_{N-1} r_{N-1} + 1 \\ r_{N-1} &= f_N \cdot 1, \end{aligned}$$

where $r_0 = p$, $r_1 = q$ and $r_1 > r_2 > \cdots > r_{N-1} > r_N = 1$. Denote by $N = N(p, q)$ the number of divisions in the Euclidian algorithm for p and q, and by $f_1, f_2, \dots, f_N$ the coefficients in the Euclidian algorithm for p and q. Define $F_i = f_1 + \cdots + f_i$ and $\epsilon(p, q) = F_N = f_1 + \cdots + f_N$, $f_1(p, q) = f_1 = [p/q]$.

Suppose that R is a 2-dimensional regular local ring dominated by V and E is a nonsingular irreducible curve on $\mathrm{Spec}\, R$. Let

$$R = R_0 \to R_1 \to R_2 \to \dots \to R_i \to \dots$$

be the sequence of quadratic transforms along ν. We denote by π_i the map $\mathrm{Spec}\, R_i \to \mathrm{Spec}\, R$ and by E_i the reduced simple normal crossing divisor $\pi_i^{-1}(E)_{red}$. We say that R_i is *free* if E_i has exactly one irreducible component. For a free ring R_i and a regular parameter $u_i \in R_i$ we will say that u_i is an exceptional coordinate if u_i is supported on E_i. A system of parameters (u_i, v_i) of a free ring R_i is called *permissible* if u_i is an exceptional parameter. The next lemma gives a description of the sequence of quadratic transforms of R along ν. See Section 6 of [5] for the proof.

Lemma 6.1. *Suppose that R is a free ring and (u, v) is a permissible system of parameters in R such that $\nu(v)/\nu(u) = p/q$ for some coprime integers p and q. Let $k = \epsilon(p, q)$, $f_1 = f_1(p, q)$ and let a and b be nonnegative integers such that $a \leq p$, $b < q$, and $aq - bp = 1$. Then the sequence of quadratic transforms along ν*

$$R = R_0 \to R_1 \to \dots \to R_{f_1} \to R_{f_1+1} \to \dots \to R_{k-1} \to R_k \tag{6.1}$$

has the following properties:

1) *$R_0, R_1, \dots, R_{f_1}$ and R_k are free rings.*
2) *Non-free rings appear in (6.1) if and only if $k > f_1$, that is, if $q \neq 1$.*
In this case $R_{f_1+1}, \dots, R_{k-1}$ are non-free.

3) *R_k has a permissible system of coordinates*

$$(X, Y) = \left(\frac{x^a}{y^b}, \frac{y^q}{x^p} - c\right),$$

where $c \in \mathbf{k}$ is the residue of y^q/x^p, and $\nu^*(X) = \nu^*(x)/q$. Moreover, $x = X^q(Y+c)^b$, $y = X^p(Y+c)^a$.

Definition 6.2. [3, 7.8] A permissible system of parameters (u, v) of a free ring R is called *admissible* if $\nu(v)$ is maximal among all regular systems of parameters containing u.

Lemma 6.3. *Suppose that (u, v) is a permissible system of parameters of a free ring R and $\nu(v)/\nu(u) = p/q$ for some coprime integers p and q. Then (u, v) are admissible if and only if $q \neq 1$.*

Proof. Assume by contradiction that $q = 1$. Then $\nu(v)/\nu(u) = p$. Denote by c the residue of vu^{-p} and set $v' = v - cu^p$. Then (u, v') are permissible parameters with $\nu(v') > \nu(v)$. So (u, v) are not admissible.

Now assume by contradiction that (u, v) are not admissible. Then there exists v' such that (u, v') are permissible parameters and $\nu(v') > \nu(v)$. By the Weierstrass Preparation theorem (Theorem 5, Section 1, Chapter VII [10]), we have that $\gamma v' = v - P(u)$, where γ is a unit, and $P \in \mathbf{k}[[u]]$ has order $n \geq 1$. Since $\nu(v') > \nu(v)$, it follows that $n\nu(u) = \nu(v)$, and so $\nu(v)/\nu(u) = n$. Thus $p = n$ and $q = 1$, a contradiction. ☐

Lemma 6.4. *Suppose that (u, v) and $(\bar{u}, \bar{v})$ are admissible systems of parameters of a free ring R. Then $\nu(u) = \nu(\bar{u})$ and $\nu(v) = \nu(\bar{v})$.*

Proof. Since u and $\bar{u}$ are supported on the same curve in $\operatorname{Spec} R$ we have $\bar{u} = u\gamma$ for some unit $\gamma \in R$; in particular $\nu(u) = \nu(\bar{u})$. The equality $\bar{u} = u\gamma$ also implies that $(\bar{u}, v)$ is a permissible system of parameters in R. Since $(\bar{u}, \bar{v})$ are admissible parameters we have $\nu(v) \leq \nu(\bar{v})$. Symmetrically, $\nu(\bar{v}) \leq \nu(v)$. Thus $\nu(v) = \nu(\bar{v})$. ☐

Remark 6.5. If the value group Γ of ν is a non-discrete subgroup of $\mathbb{Q}$, it follows from [3, 7.7] that an admissible system of parameters of R always exists. If Γ is a discrete subgroup of $\mathbb{Q}$, after performing a sequence of quadratic transforms along ν we may assume that the value of the exceptional parameter u generates Γ. It follows from Lemma 6.3 that R does not have an admissible system of parameters.

We now fix regular parameters (u, v) of R and assume that (u, v) is a permissible system of parameters in R by setting E to be the curve on $\operatorname{Spec} R$ defined by $u = 0$. Recall the definition of $\bar{p}_l$ and $\bar{q}_l$ given in Section 4. In what follows, for all $l > 0$ let a_l and b_l be nonnegative integers such that $a_l\bar{q}_l - b_l\bar{p}_l = 1$ and $a_l \leq \bar{p}_l$, $b_l < \bar{q}_l$. Let $\bar{k}_0 = 0$ and $\bar{k}_l = \bar{k}_{l-1} + \epsilon(\bar{p}_l, \bar{q}_l)$. Also set $k_0 = 0$ and $k_i = k_{i-1} + \epsilon(p_i, q_i)$ if $i > 0$.

The next theorem describes the images of independent jumping polynomials under blowups of R along ν.

Theorem 6.6. *With notations as above, let* $R = R_0 \to R_1 \to \ldots \to R_{\bar{k}_1-1} \to R_{\bar{k}_1} \to R_{\bar{k}_1+1} \to \ldots \to R_{\bar{k}_l} \to \ldots$ *be the sequence of quadratic transforms along* ν*. Assume that the value group of* ν *is a non-discrete subgroup of* $\mathbb{Q}$*. Then for all* $l \geq 0$*,* $R_{\bar{k}_l}$ *is free and has an admissible system of parameters* (u_l, v_l) *such that*

1) u_l *is an exceptional parameter and* $\nu(u_l) = 1/\bar{Q}_l$*,*
2) $v_l = H_{l+1}/\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}}$ *is the strict transform of* H_{l+1} *in* $R_{\bar{k}_l}$ *and* $\nu(v_l) = (1/\bar{Q}_l) \cdot (\bar{p}_{l+1}/\bar{q}_{l+1})$*,*
3) *for all* $0 \leq j \leq l$ *there exists a unit* $\gamma_{j,l} \in R_{\bar{k}_l}$ *such that* $H_j = u_l^{\bar{Q}_l \bar{\beta}_j} \gamma_{j,l}$*.*

Proof. We first show that 3) implies that for all $1 \leq m \leq l+1$ and for all $i_l < i' < i_{l+1}$ there exist units $\tau_{m,l}$ and $\tau'_{i',l}$ in $R_{\bar{k}_l}$ such that

$$\prod_{j=0}^{m-1} H_j^{n_{i_m,i_j}} = u_l^{\bar{Q}_l \bar{q}_m \bar{\beta}_m} \tau_{m,l} \quad \text{and} \quad \prod_{j=0}^{l} H_j^{n_{i',i_j}} = u_l^{\bar{Q}_l \beta_{i'}} \tau'_{i',l}.$$

Indeed, the following lines of equalities hold

$$\prod_{j=0}^{m-1} H_j^{n_{i_m,i_j}} = \prod_{j=0}^{m-1} (u_l^{\bar{Q}_l \bar{\beta}_j} \gamma_{j,l})^{n_{i_m,i_j}} = u_l^{\bar{Q}_l \sum_{j=0}^{m-1} n_{i_m,i_j} \bar{\beta}_j} \prod_{j=0}^{m-1} \gamma_{j,l}^{n_{i_m,i_j}} = u_l^{\bar{Q}_l \bar{q}_m \bar{\beta}_m} \tau_{m,l}$$

and

$$\prod_{j=0}^{l} H_j^{n_{i',i_j}} = \prod_{j=0}^{l} (u_l^{\bar{Q}_l \bar{\beta}_j} \gamma_{j,l})^{n_{i',i_j}} = u_l^{\bar{Q}_l \sum_{j=0}^{l} n_{i',i_j} \bar{\beta}_j} \prod_{j=0}^{l} \gamma_{j,l}^{n_{i',i_j}} = u_l^{\bar{Q}_l q_{i'} \beta_{i'}} \tau'_{i',l} = u_l^{\bar{Q}_l \beta_{i'}} \tau'_{i',l}.$$

We will prove the theorem by induction on l. For $l = 0$ the system of parameters $(u_0, v_0) = (u, H_1)$ in R clearly satisfies all the conclusions. Assume now that the statement holds for $l-1$. Then by Lemma 6.1 the ring $R_{\bar{k}_l}$ is free and has a permissible system of parameters (u_l, z_l) such that

$$u_l = \frac{u_{l-1}^{a_l}}{v_{l-1}^{b_l}}, \quad \nu(u_l) = \frac{1}{\bar{Q}_l} \quad \text{and} \quad z_l = \frac{v_{l-1}^{\bar{q}_l}}{u_{l-1}^{\bar{p}_l}} - c_l,$$

where $c_l \in \mathbf{k} - \{0\}$ is the residue of $v_{l-1}^{\bar{q}_l} u_{l-1}^{-\bar{p}_l}$. It follows that u_l satisfies conclusion 1). We will now show that u_l also satisfies conclusion 3).

If $j \leq l-1$ then

$$H_j = u_{l-1}^{\bar{Q}_{l-1} \bar{\beta}_j} \gamma_{j,l-1} = (u_l^{\bar{q}_l} (z_l + c_l)^{b_l})^{\bar{Q}_{l-1} \bar{\beta}_j} \gamma_{j,l-1} = u_l^{\bar{Q}_l \bar{\beta}_j} \gamma_{j,l}$$

and

$$H_l = v_{l-1} \prod_{j=0}^{l-2} H_j^{n_{i_{l-1},i_j}} = v_{l-1} u_{l-1}^{\bar{Q}_{l-1} \bar{q}_{l-1} \bar{\beta}_{l-1}} \tau_{l-1,l-1} =$$

$$= u_l^{\bar{p}_l} u_l^{\bar{Q}_l \bar{q}_{l-1} \bar{\beta}_{l-1}} (z_l + c_l)^{a_l + b_l \bar{Q}_{l-1} \bar{q}_{l-1} \bar{\beta}_{l-1}} \tau_{l-1,l-1} = u_l^{\bar{Q}_l \bar{\beta}_l} \gamma_{l,l},$$

where $\gamma_{j,l}$ and $\gamma_{l,l}$ are units in $R_{\bar{k}_l}$.

Let us set $v_l = H_{l+1}/\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}}$. Then $\nu(v_l) = \bar\beta_{l+1} - \bar q_l \bar\beta_l = (1/\bar Q_l)\cdot(\bar p_{l+1}/\bar q_{l+1})$.

It only remains to show that (u_l, v_l) is an admissible system of parameters in $R_{\bar k_l}$. To this end we will present v_l in terms of u_l and z_l. We have that

$$v_l = \frac{H_l^{\bar q_l}}{\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}}} - \lambda_{i_l}\delta_{i_l} - \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'}\delta_{i'} \frac{\prod_{j=0}^{l} H_j^{n_{i',i_j}}}{\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}}}.$$

Denote by τ_l the unit $\tau_{l-1,l-1}^{\bar q_l}\tau_{l,l-1}^{-1}$ in $R_{\bar k_{l-1}}$ and denote by t_l the residue of τ_l. Then the following line of equalities holds

$$\frac{H_l^{\bar q_l}}{\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}}} = \frac{v_{l-1}^{\bar q_l} u_{l-1}^{\bar Q_l \bar q_{l-1}\bar\beta_{l-1}} \tau_{l-1,l-1}^{\bar q_l}}{u_{l-1}^{\bar Q_{l-1}\bar q_l \bar\beta_l}\tau_{l,l-1}} = \frac{v_{l-1}^{\bar q_l}}{u_{l-1}^{\bar p_l}}\tau_l = (z_l + c_l)\tau_l = z_l\tau_l + c_l t_l + u_l w_l,$$

where $w_l \in R_{\bar k_l}$ is such that $u_l w_l = c_l(\tau_l - t_l)$, and the existence of w_l is guaranteed by the inclusion $(\tau_l - t_l) \in m_{R_{\bar k_{l-1}}} R_{\bar k_{l-1}} \subset u_l R_{\bar k_l}$. We also notice that taking the residues of both sides of this equality gives $\lambda_{i_l} = c_l t_l$. Furthermore, since $(\delta_{i_l} - 1) \in m_R R \subset u_l R_{\bar k_l}$ we have $\lambda_{i_l}\delta_{i_l} = \lambda_{i_l} + u_l w_l'$ for some $w_l' \in R_{\bar k_l}$.

For $i_l < i' < i_{l+1}$ denote by $P_{i'}$ the positive integer $\bar Q_l(\beta_{i'} - \bar q_l\bar\beta_l) = p_{i_l+1} + p_{i_l+2} + \cdots + p_{i'}$. Then

$$\sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'}\delta_{i'} \frac{\prod_{j=0}^{l} H_j^{n_{i',i_j}}}{\prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}}} = \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'}\delta_{i'} \frac{u_l^{\bar Q_l \beta_{i'}}\tau'_{i',l}}{u_l^{\bar Q_l \bar q_l \bar\beta_l}\tau_{l,l}} = \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'}\delta_{i'}\tau'_{i',l}\tau_{l,l}^{-1} u_l^{P_i'}$$

$$= u_l \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'}\delta_{i'}\tau'_{i',l}\tau_{l,l}^{-1} u_l^{P_i'-1} = u_l w_l'',$$

where $w_l'' \in R_{\bar k_l}$. In particular, $w_l'' = 0$ if $i_{l+1} = i_l + 1$.

Combining all the above we get $v_l = z_l\tau_l + u_l w_l - u_l w_l' - u_l w_l''$. Thus (u_l, v_l) form a system of regular parameters in $R_{\bar k_l}$. Moreover, (u_l, v_l) is an admissible system by Lemma 6.3, since $\nu(v_l)/\nu(u_l) = \bar p_{l+1}/\bar q_{l+1}$ and $\bar q_{l+1} \neq 1$. This completes the proof of the theorem. □

Using the same line of arguments with H_l replaced by T_i and $\bar k_l$ replaced by k_i, we obtain the following property of the sequence of jumping polynomials in R.

Remark 6.7. (see also [5, 7.5]). Let $R = R_0 \to R_1 \to \ldots \to R_{k_1-1} \to R_{k_1} \to R_{k_1+1} \to \ldots \to R_{k_i} \to \ldots$ be the sequence of quadratic transforms along ν. Then for all $i > 0$, R_{k_i} is free and has a system of regular parameters (u_i, v_i) such that

1) u_i is an exceptional parameter and $\nu(u_i) = 1/Q_i$,
2) $v_i = T_{i+1}/\prod_{j=0}^{i-1} T_j^{n_{i,j}}$ is the strict transform of T_{i+1} in R_{k_i} and $\nu(v_i) = (1/Q_i)\cdot(p_{i+1}/q_{i+1})$,
3) for all $0 \le j \le i$ there exists a unit $\gamma_{j,i} \in R_{k_i}$ such that $T_j = u_i^{Q_i\beta_j}\gamma_{j,i}$.

Remark 6.8. For every $l \geq 0$ we have that $\bar{k}_l = k_{i_l}$. This is trivial for $l = 0$. By induction on l, assume that $\bar{k}_{l-1} = k_{i_{l-1}}$. We have that $\epsilon(\bar{p}_l, \bar{q}_l) = \epsilon((p_{i_{l-1}+1} + \cdots + p_{i_l-1})q_{i_l} + p_{i_l}, q_{i_l}) = p_{i_{l-1}+1} + \cdots + p_{i_l-1} + \epsilon(p_{i_l}, q_{i_l}) = \epsilon(p_{i_{l-1}+1}, q_{i_{l-1}+1}) + \cdots + \epsilon(p_{i_l-1}, q_{i_l-1}) + \epsilon(p_{i_l}, q_{i_l})$, since $q_{i_{l-1}+1} = \cdots = q_{i_l-1} = 1$. Therefore $\bar{k}_l = \bar{k}_{l-1} + \epsilon(\bar{p}_l, \bar{q}_l) = k_{i_{l-1}} + \epsilon(p_{i_{l-1}+1}, q_{i_{l-1}+1}) + \cdots + \epsilon(p_{i_l-1}, q_{i_l-1}) + \epsilon(p_{i_l}, q_{i_l}) = k_{i_l}$.

Remark 6.9. Notice that (u_i, v_i) of Remark 6.7 are not in general admissible parameters of R_{k_i}. If the index k_i corresponds to an admissible choice of parameters then $i + 1 = i_l$ for some l, that is, $k_i = k_{i_l-1} = \bar{k}_l - \epsilon(p_{i_l}, q_{i_l})$.

7. Strong Monomialization

In this section we recall definitions and results from Section 7 of [3] that will be needed in this paper.

Let $\mathbf{k}$ be an algebraically closed field of characteristic $\mathbf{p} > 0$, and let K^*/K be a finite and separable extension of algebraic function fields of transcendence degree two over $\mathbf{k}$. Let ν^* be a $\mathbf{k}$-valuation of K^* with valuation ring V^* and value group Γ^*. Let ν be the restriction of ν^* to K with valuation ring V and value group Γ. Consider an extension of algebraic regular local rings $R \subset S$ where R has quotient field K, S has quotient field K^*, R is dominated by S and S is dominated by V^*. We assume that Γ^* and Γ are non-discrete subgroups of $\mathbb{Q}$. In particular we have that $\operatorname{trdeg}_{\mathbf{k}}(V^*/m_{V^*}) = 0$, and so $V^*/m_{V^*} \simeq \mathbf{k}$, since $\mathbf{k}$ is algebraically closed. Let $S = S_0 \to S_1 \to S_2 \to \ldots \to S_s \to \ldots$ be the quadratic sequence along ν^*, and let $R = R_0 \to R_1 \to R_2 \to \ldots \to R_r \to \ldots$ be the quadratic sequence along ν. For $i \geq 1$ we denote the reduced exceptional locus of $\operatorname{Spec} R_i \to \operatorname{Spec} R$ by E_i, and the reduced exceptional locus of $\operatorname{Spec} S_i \to \operatorname{Spec} S$ by F_i.

Definition 7.1. Given a pair (r, s) of positive integers, the pair (R_r, S_s) is said to be *prepared* if the following properties hold:

i) S_s dominates R_r.
ii) R_r and S_s are free.
iii) The critical locus of $\operatorname{Spec} S_s \to \operatorname{Spec} R_r$ is contained in F_s.
iv) We have $u = x^t\delta$, where u (resp. x) is a regular parameter of R_r (resp. S_s) whose support is E_r (resp. F_s), and δ is a unit in S_s.

It is shown in [3, 7.6] that given a prepared pair (R_r, S_s), any pair $(R_{r'}, S_{s'})$ with $r' \geq r$, $s' \geq s$, and such that both $R_{r'}$ and $S_{s'}$ are free and $S_{s'}$ dominates $R_{r'}$, is also prepared.

Let (R_r, S_s) be prepared. Recall that a regular system of parameters (r.s.p.) (u, v) of R_r is said to be admissible if the support of u is equal to E_r and if $\nu(v)$ is maximal among all such r.s.p. containing u.

Now we briefly recall the algorithm described in Section 7.4 of [3]. We fix a prepared pair $(R, S) =: (R_{r_0}, S_{s_0})$ such that $m_R S$ is not a principal ideal. By induction on $n \geq 0$, we associate with a given prepared pair (R_{r_n}, S_{s_n}) such that $m_{R_{r_n}} S_{s_n}$ is not a principal ideal, a new prepared pair $(R_{r_{n+1}}, S_{s_{n+1}})$, with $r_{n+1} > r_n$, $s_{n+1} > s_n$, and such that $m_{R_{r_{n+1}}} S_{s_{n+1}}$ is not a principal ideal.

Let (u_{r_n}, v_{r_n}) be an admissible r.s.p. of R_{r_n}. Let (x_{s_n}, y_{s_n}) be a r.s.p. of S_{s_n} such that the support of x_{s_n} is F_{s_n}. Write

$$\begin{aligned} u_{r_n} &= x_{s_n}^{t_n} \delta_n \\ v_{r_n} &= x_{s_n}^{\beta_n} f_n \end{aligned} \tag{7.1}$$

where δ_n is a unit in S_{s_n}, and x_{s_n} does not divide f_n. Notice that f_n is not a unit, since $m_{R_{r_n}} S_{s_n}$ is not a principal ideal. Among all $s \geq s_n$, there is a least integer s_{n+1} such that $S_{s_{n+1}}$ is free and the strict transform of $\mathrm{div}(f_n)$ in $S_{s_{n+1}}$ is empty. It follows that $s_{n+1} > s_n$. By construction, $m_{R_r} S_{s_{n+1}}$ is a principal ideal. The nonempty set of integers $r > r_n$ such that $S_{s_{n+1}}$ dominates R_{r_n} has a maximal element denoted by r_{n+1}. This completes the definition of the pair $(R_{r_{n+1}}, S_{s_{n+1}})$. It is shown in [3, 7.18] that the algorithm is well defined, that is, the pair (r_{n+1}, s_{n+1}) does not depend on the choice of an admissible r.s.p. (u_{r_n}, v_{r_n}) of R_{r_n}. Moreover, $(R_{r_{n+1}}, S_{s_{n+1}})$ is prepared.

We are now ready to state Cutkosky and Piltant's Strong Monomialization theorem for defectless extensions.

Theorem 7.2. (Strong Monomialization [3, 7.35]) *In the above set-up and notations, assume that V^*/V is defectless. The inclusion $R_{r_n} \subset S_{s_n}$ is given for $n >> 0$ by*

$$\begin{aligned} u_{r_n} &= x_{s_n}^{t} \delta_n \\ v_{r_n} &= y_{s_n} \end{aligned} \tag{7.2}$$

where t is a positive integer, δ_n is a unit in S_{s_n}, and (x_{s_n}, y_{s_n}) is an admissible r.s.p. of S_{s_n}.

In particular the theorem shows that the equation defining the inclusion $R_{r_n} \subset S_{s_n}$ gets a stable form. We recall that Strong Monomialization may not hold if the extension V^*/V has a defect [3, 7.38]. It is not known if the weaker form of the monomialization theorem [3, 4.1] holds in this case.

8. Quadratic transforms

In this section we discuss several preparatory results that we will need in the proof of the main theorem.

Throughout this section let $R \subset S$ be an extension of two dimensional algebraic regular local rings with quotient fields K and K^* respectively, such that V^* dominates S and S dominates R. We further assume that Γ^* and Γ are non-discrete subgroups of $\mathbb{Q}$. Suppose that R has regular parameters (u, v) and S has regular parameters (x, y). Let p and q be positive coprime integers such that $\nu(v)/\nu(u) = p/q$ and let $k = \epsilon(p, q)$. Let p' and q' be positive coprime integers such that $\nu^*(y)/\nu^*(x) = p'/q'$ and let $k' = \epsilon(p', q')$.

Remark 8.1. Suppose that the inclusion $R \subset S$ is given by

$$\begin{aligned} u &= x^t\delta \\ v &= y, \end{aligned} \tag{8.1}$$

where t is a positive integer and δ is a unit in S. We have that $\nu(v)/\nu(u) = \nu^*(y)/t\nu^*(x) = p'/tq'$. Suppose that (x, y) are admissible parameters. Then by Lemma 6.3 (u, v) are admissible, since $q' \neq 1$ implies $q \neq 1$. Viceversa, if (u, v) are admissible and t divides p', then (x, y) are admissible, since in this case $q' = q$.

Remark 8.2. Suppose that the inclusion $R \subset S$ is given by

$$\begin{aligned} u &= x^t\delta \\ v &= y, \end{aligned} \tag{8.2}$$

where t is a positive integer and δ is a unit in S. Let g be the greatest common divisor of t and p'. Recall that $\nu(v)/\nu(u) = \nu^*(y)/t\nu^*(x) = p'/tq'$. Writing $t = g\tilde{t}$ and $p' = g\tilde{p}$, where $(\tilde{t}, \tilde{p}) = 1$, gives $p = \tilde{p}$ and $q = q'\tilde{t}$. After possibly multiplying u by a constant we may assume that $\delta = 1 + w$ for some $w \in m_S$. Let a, b, a', b' be nonnegative integers such that $a \leq p$, $a' \leq p'$, $b < q$, $b' < q'$ and $aq - bp = 1$, $a'q' - b'p' = 1$.

By Lemma 6.1 applied to S and R respectively, we get that $S_{k'}$ has a permissible system of parameters $(X, Y') = \left(x^{a'}/y^{b'}, y^{q'}/x^{p'} - c'\right)$, where $c' \in \mathbf{k}$ is the residue of $y^{q'}/x^{p'}$, and R_k has a permissible system of parameters $(U, V) = \left(u^a/v^b, v^q/u^p - c\right)$, where $c \in \mathbf{k}$ is the residue of v^q/u^p. Moreover, $x = X^{q'}(Y' + c')^{b'}$, $y = X^{p'}(Y' + c')^{a'}$ and $u = U^q(V + c)^b$, $v = U^p(V + c)^a$.

Now

$$U = \frac{u^a}{v^b} = \frac{x^{ta}\delta^a}{y^b} = \frac{[X^{q'}(Y' + c')^{b'}]^{ta}\delta^a}{[X^{p'}(Y' + c')^{a'}]^b} = \frac{X^{q'ta}}{X^{p'b}}\delta^a(Y' + c')^{b'ta - a'b} = X^g\Delta,$$

where $\Delta = \delta^a(Y' + c')^{b'ta - a'b}$ is a unit in $S_{k'}$. Notice that the last equality holds since $q'ta - p'b = q'g\tilde{t}a - g\tilde{p}b = g(q'\tilde{t}a - \tilde{p}b) = g(qa - pb) = g$.

Furthermore notice that

$$\frac{v^q}{u^p} = \frac{y^q}{x^{tp}}\delta^{-p} = \frac{y^{q'\tilde{t}}}{x^{g\tilde{t}p}}\delta^{-p} = \left(\frac{y^{q'}}{x^{p'}}\right)^{\tilde{t}}\delta^{-p} = \left(\frac{y^{q'}}{x^{p'}}\right)^{\tilde{t}}(1 + w)^{-p}.$$

Therefore $c = (c')^{\tilde{t}}$, and

$$V = \frac{v^q}{u^p} - c = \left(\frac{y^{q'}}{x^{p'}}\right)^{\tilde{t}}(1 + w)^{-p} - (c')^{\tilde{t}} = \left(\frac{y^{q'}}{x^{p'}}\right)^{\tilde{t}} - (c')^{\tilde{t}} + W$$

where $W \in wS \subset m_S$. Since $m_S \subset (X)S_{k'}$ we have that $W = XZ$ for some $Z \in S_{k'}$. Write $\tilde{t} = \mathbf{p}^n t'$, where $n \geq 0$ and $\mathbf{p}$ does not divide t'.

Then

$$V = \left[\left(\frac{y^{q'}}{x^{p'}}\right)^{t'} - (c')^{t'}\right]^{\mathbf{p}^n} + XZ = \left(\frac{y^{q'}}{x^{p'}} - c'\right)^{\mathbf{p}^n} \gamma^{\mathbf{p}^n} + XZ = (Y')^{\mathbf{p}^n}\gamma_1 + XZ,$$

where

$$\gamma = \frac{\left(\frac{y^{q'}}{x^{p'}}\right)^{t'} - (c')^{t'}}{\left(\frac{y^{q'}}{x^{p'}} - c'\right)}$$

and $\gamma_1 = \gamma^{\mathbf{p}^n}$ is a unit in $S_{k'}$. In particular $S_{k'}$ dominates R_k.

Remark 8.3. In the set-up of Remark 8.2 we will be interested in the case when $(t, q) = 1$, or equivalently t divides p'. In this case $g = t$, $\tilde{t} = 1$, $t' = 1$ and $n = 0$. In particular we have that $V = Y'\gamma_1 + XZ$. Setting $Y = Y'\gamma_1 + XZ$, we have that (X, Y) is a permissible system of parameters of $S_{k'}$, and the inclusion $R_k \subset S_{k'}$ is given by

$$\begin{aligned} U &= X^t\Delta \\ V &= Y. \end{aligned} \tag{8.3}$$

Remark 8.4. Suppose that the inclusion $R = R_{r_0} \subset S = S_{s_0}$ is given by

$$\begin{aligned} u &= x^t\delta \\ v &= y, \end{aligned} \tag{8.4}$$

where t is a positive integer and δ is a unit in S. Suppose also that (u, v) are admissible parameters of R. We claim that $(R_k, S_{k'}) = (R_{r_1}, S_{s_1})$. By definition (see Section 7), S_{s_1} is the first free ring in the quadratic sequence for S such that the strict transform of y in such ring is empty. The sequence $S = S_0 \to S_1 \to \ldots \to S_{k'}$ has been explicitly described in Section 6 of [5] (see also Lemma 6.1). It follows from this description that $S_{s_1} = S_{k'}$. Furthermore, by Remark 8.2 $S_{k'}$ dominates R_k, and R_k is the biggest free ring in the quadratic sequence for R with this property. So $R_k = R_{r_1}$.

As in Section 4, for all $i > 0$ let (p_i, q_i) be the pair of coprime integers defined in the construction of jumping polynomials $\{T_i\}_{i\geq 0}$ in R. Let $\{T_{i_l}\}_{l\geq 0}$ be the sequence of independent jumping polynomials in R. For $l > 0$ let $\bar{q}_l = q_{i_l}$, $\bar{p}_l = (p_{i_{l-1}+1} + \cdots + p_{i_l-1})\bar{q}_l + p_{i_l}$. Let $\bar{k}_0 = 0$, $\bar{k}_l = \bar{k}_{l-1} + \epsilon(\bar{p}_l, \bar{q}_l)$ if $l > 0$. Let $k_0 = 0$ and $k_i = k_{i-1} + \epsilon(p_i, q_i)$ if $i > 0$. For all $i > 0$ let (p'_i, q'_i) be the pair of coprime integers defined in the construction of jumping polynomials $\{T'_i\}_{i\geq 0}$ in S. Let $\{T'_{j_l}\}_{l\geq 0}$ be the sequence of independent jumping polynomials in S. For $l > 0$ let $\bar{q}'_l = q'_{j_l}$, $\bar{p}'_l = (p'_{j_{l-1}+1} + \cdots + p'_{j_l-1})\bar{q}'_l + p'_{j_l}$. Let $\bar{k}'_0 = 0$, $\bar{k}'_l = \bar{k}'_{l-1} + \epsilon(\bar{p}'_l, \bar{q}'_l)$ if $l > 0$. Finally, let $k'_0 = 0$, and $k'_i = k'_{i-1} + \epsilon(p'_i, q'_i)$ if $i > 0$.

Lemma 8.5. *Assume that the equation defining the inclusion $R_{r_l} \subset S_{s_l}$ for $l \geq 0$ has the stable form of Theorem 7.2. Then $R_{r_l} = R_{\bar{k}_l} = R_{k_{i_l}}$, and $S_{s_l} = S_{\bar{k}'_l} = S_{k'_{j_l}}$. In particular, if (u_{r_l}, v_{r_l}) is an admissible system of parameters of R_{r_l} we have that $\nu(v_{r_l})/\nu(u_{r_l}) = \bar{p}_{l+1}/\bar{q}_{l+1}$. Similarly, if (x_{s_l}, y_{s_l}) is an admissible system of parameters of S_{s_l} we have that $\nu^*(y_{s_l})/\nu^*(x_{s_l}) = \bar{p}'_{l+1}/\bar{q}'_{l+1}$.*

Proof. It suffices to show that for every $l \geq 0$ we have that $R_{r_l} = R_{\bar{k}_l}$ and $S_{s_l} = S_{\bar{k}'_l}$. Then the other claims follow from Remark 6.8, Theorem 6.6 and Lemma 6.4.

We apply induction on l. The case $l = 0$ is trivial since by definition $R = R_{r_0} = R_{\bar{k}_0}$ and $S = S_{s_0} = S_{\bar{k}'_0}$. Now assume that $R_{r_{l-1}} = R_{\bar{k}_{l-1}}$ and $S_{s_{l-1}} = S_{\bar{k}'_{l-1}}$, that is, $r_{l-1} = \bar{k}_{l-1}$ and $s_{l-1} = \bar{k}'_{l-1}$. By Remark 8.4 applied to the inclusion $R_{r_{l-1}} \subset S_{s_{l-1}}$, we have that $r_l = r_{l-1} + \epsilon(\bar{p}_l, \bar{q}_l) = \bar{k}_{l-1} + \epsilon(\bar{p}_l, \bar{q}_l) = \bar{k}_l$ and $s_l = s_{l-1} + \epsilon(\bar{p}'_l, \bar{q}'_l) = \bar{k}'_{l-1} + \epsilon(\bar{p}'_l, \bar{q}'_l) = \bar{k}'_l$. □

9. Monomialization of generating sequences

The goal of this section is to prove the following theorem.

Theorem 9.1. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* $\mathbf{p} > 0$*, and let* K^*/K *be a finite separable extension of algebraic function fields of transcendence degree* 2 *over* $\mathbf{k}$*. Let* ν^* *be a* $\mathbf{k}$*-valuation of* K^**, with valuation ring* V^* *and value group* Γ^**, and let* ν *be the restriction of* ν^* *to* K*, with valuation ring* V *and value group* Γ*. Assume that* V^*/V *is defectless. Suppose that* $R \subset S$ *is an extension of algebraic regular local rings with quotient fields* K *and* K^* *respectively, such that* V^* *dominates* S *and* S *dominates* R*. Then there exist sequences of quadratic transforms* $R \to \bar{R}$ *and* $S \to \bar{S}$ *along* ν^* *such that* $\bar{S}$ *dominates* $\bar{R}$ *and the map between generating sequences of* ν *and* ν^* *in* $\bar{R}$ *and* $\bar{S}$ *respectively, has a toroidal structure.*

Proof. By the discussion of Section 3 we only need to consider the case when Γ^* is a subgroup of $\mathbb{Q}$ and $\mathrm{trdeg}_{\mathbf{k}}(V^*/m_{V^*}) = 0$.

By [3, 7.3] if Γ^* and Γ are discrete, and by Theorem 7.2 if Γ^* and Γ are non-discrete, we may assume that R has a regular system of parameters (u, v), S has a regular system of parameters (x, y) such the inclusion $R \subset S$ is given by

$$\begin{aligned} u &= x^t \delta \\ v &= y, \end{aligned} \tag{9.1}$$

where t is a positive integer and δ is a unit in S. After possibly multiplying u by a constant we may assume that $\delta = 1 + w$ for some $w \in m_S$. If $t = 1$ then the conclusion of the theorem is trivial, so assume that $t > 1$. We may also assume that Γ is normalized so that $\nu(u) = 1$.

As in Section 4, let $\{T_i\}_{i \geq 0}$ be the sequence of jumping polynomials in R corresponding to (u, v) and the trivial sequence of units $\{1\}_{i>0}$. For all $i > 0$ let the coprime integers p_i and q_i be defined as in the construction of $\{T_i\}_{i \geq 0}$. Let $\{n_{i,0}\}_{i>0}$ be the powers defined in the construction of $\{T_i\}_{i \geq 0}$. For $l > 0$ let the coprime integers $\bar{p}_l$ and $\bar{q}_l$ be defined as in the construction of the subsequence $\{T_{i_l}\}_{l \geq 0} = \{H_l\}_{l \geq 0}$ of independent jumping polynomials.

Similarly, let $\{T'_i\}_{i \geq 0}$ be the sequence of jumping polynomials in S corresponding to (x, y) and the sequence of units $\{\delta^{n_{i,0}}\}_{i>0}$. For all $i > 0$ let the coprime integers p'_i and q'_i be defined as in the construction of $\{T'_i\}_{i \geq 0}$. For $l > 0$ let the coprime integers

$\bar{p}'_l$ and $\bar{q}'_l$ be defined as in the construction of the subsequence $\{H'_l\}_{l\geq 0}$ of independent jumping polynomials.

First, let us assume that Γ^* and Γ are discrete subgroups of $\mathbb{Q}$. After performing a sequence of quadratic transforms along ν and normalizing Γ we may further assume that $\nu(u) = 1$ generates Γ. Since $q_i = 1$ for all $i > 0$, by Theorem 5.9 we have that $T'_i = T_i$ for all $i > 0$. Then by Theorem 5.4 $\{T_i\}_{i\geq 0}$ is a generating sequence in R and $\{T'_i\}_{i\geq 0} = \{x, \{T_i\}_{i>0}\}$ is a generating sequence in S. Notice also that Γ^* is generated by the set of values $\{\nu^*(T'_i)\}_{i\geq 0}$, and that for any $i > 0$ the value $\nu^*(T'_i) = \nu(T_i)$ is an integer. Thus $\nu^*(x) = \nu(u)/t = 1/t$ generates Γ^*. This concludes the proof of the theorem in the discrete case.

Now let us assume that Γ^* and Γ are non-discrete subgroups of $\mathbb{Q}$. By Theorem 7.2 we may assume that equation (9.1) is stable and that $(u, v) = (u_{r_0}, v_{r_0})$, $(x, y) = (x_{s_0}, y_{s_0})$ are admissible parameters of $R = R_{r_0}$, $S = S_{s_0}$ respectively.

First, let us assume that $(t, Q_k) = 1$ for all $k > 0$. By Theorem 5.9 we have that $T'_i = T_i$ for all $i > 0$. Since $q'_i = q_i$ for all $i > 0$, we have that T'_i is an independent jumping polynomial in S if and only if T_i is an independent jumping polynomial in R. If follows that $\{H'_l\}_{l\geq 0} = \{x, \{H_l\}_{l>0}\}$.

By Theorem 5.7 $\{H_l\}_{l\geq 0}$ is a generating sequence in R and $\{x, \{H_l\}_{l>0}\}$ is a generating sequence in S. Moreover, $\{x, \{H_l\}_{l>0}\}$ is a minimal generating sequence of ν^* since $\bar{p}'_1 \geq p'_1 = tp_1 > 1$, since $t > 1$. The conclusion of the theorem follows.

Otherwise let M be the integer such that $(t, Q_k) = 1$ for all $0 \leq k < M$ and $(t, q_M) \neq 1$. Since $(t, q_M) \neq 1$ it follows that $q_M \neq 1$, and so $M = i_l$ for some l. The inclusion $R_{r_{l-1}} \subset S_{s_{l-1}}$ is given by the stable monomial form

$$\begin{aligned} u_{r_{l-1}} &= x^t_{s_{l-1}}\delta_{l-1} \\ v_{r_{l-1}} &= y_{s_{l-1}} \end{aligned} \tag{9.2}$$

where δ_{l-1} is a unit in $S_{s_{l-1}}$, $(x_{s_{l-1}}, y_{s_{l-1}})$ are admissible parameters of $S_{s_{l-1}}$, and $(u_{r_{l-1}}, v_{r_{l-1}})$ are admissible parameters of $R_{r_{l-1}}$.

By Lemma 8.5 we have that $\nu(v_{r_{l-1}})/\nu(u_{r_{l-1}}) = \bar{p}_l/\bar{q}_l$, and $\nu^*(y_{s_{l-1}})/\nu^*(x_{s_{l-1}}) = \bar{p}'_l/\bar{q}'_l$. Recall that $\bar{q}_l = q_{i_l} = q_M$. We have that $(t, \bar{q}_l) \neq 1$, or equivalently t does not divide $\bar{p}'_l$. Now we apply Remark 8.2 and Remark 8.4 with R and S replaced by $R_{r_{l-1}}$ and $S_{s_{l-1}}$ respectively.

It follows that R_{r_l} has permissible regular parameters (u_{s_l}, v_{s_l}) and S_{s_l} has permissible regular parameters (x_{s_l}, y_{s_l}) such that $u_{s_l} = x^g_{s_l}\Delta$, where Δ is a unit in S_{s_l} and $g = (\bar{p}'_l, t) < t$. Hence we obtain a contradiction to the stable form of equation (9.1).

We then conclude that $(t, Q_k) = 1$ for all $k > 0$. □

Remark 9.2. In the proof of Theorem 9.1 we have $\{H_l\}_{l\geq 0}$ is a minimal generating sequence of ν in R if $\bar{p}_1 \neq 1$, or equivalently, $\bar{p}'_1 \neq t$. Otherwise $\{H_l\}_{l>0}$ is a minimal generating sequence of ν in R.

Recall that the integers k_i are defined as $k_0 = 0$ and $k_i = k_{i-1} + \epsilon(p_i, q_i)$ if $i > 0$. Similarly $k'_0 = 0$ and $k'_i = k'_{i-1} + \epsilon(p'_i, q'_i)$ for all $i > 0$.

Corollary 9.3. *In the set-up of Theorem 9.1, assume that the inclusion $R \subset S$ is given by the stable form (9.1). The sequences of quadratic transforms*

$$R = R_0 \to R_{k_1} \to \ldots \to R_{k_{i-1}} \to R_{k_i} \to \ldots$$

and

$$S = S_0 \to S_{k'_1} \to \ldots \to S_{k'_{i-1}} \to S_{k'_i} \to \ldots$$

have the following properties: for all $i \geq 0$ the rings R_{k_i} and $S_{k'_i}$ are free, there exist permissible systems of parameters (u_i, v_i) in R_{k_i} and (x_i, y_i) in $S_{k'_i}$ and a unit $\delta_i \in S_{k'_i}$ such that

$$\begin{aligned} u_i &= x_i^t \delta_i \\ v_i &= y_i, \end{aligned} \tag{9.3}$$

and $\nu^*(y_i)/\nu^*(x_i) = p'_{i+1}/q'_{i+1}$.

Proof. The proof of Theorem 9.1 shows that for all $i \geq 1$ we have $(t, q_i) = 1$, $q'_i = q_i$, and $p'_i = tp_i$. The conclusion is trivial for $i = 0$. Assume that $i > 0$ and that the statement holds for $i - 1$. We apply Remark 8.3 to $R_{k_{i-1}} \subset S_{k'_{i-1}}$. Notice that $\nu^*(y_{i-1})/\nu^*(x_{i-1}) = p'_i/q'_i$ and $\nu(v_{i-1})/\nu(u_{i-1}) = p'_i/tq'_i = p_i/q_i$, and therefore $k = \epsilon(p_i, q_i)$ and $k' = \epsilon(p'_i, q'_i)$. Thus R_{k_i} and $S_{k'_i}$ are free rings and there exist permissible systems of regular parameters (u_i, w_i) in R_{k_i} and (x_i, z_i) in $S_{k'_i}$ such that $u_i = x_i^t \delta_i$ and $w_i = z_i$ for some unit $\delta_i \in S_{k'_i}$.

Now by Remark 6.7 we get that R_{k_i} has a system of regular parameters (h_i, v_i) such that h_i is an exceptional parameter, $\nu(h_i) = 1/Q_i$ and $\nu(v_i) = (1/Q_i) \cdot (p_{i+1}/q_{i+1})$. Since u_i is also an exceptional parameter in R_{k_i} we have $u_i = h_i \gamma$ for some unit $\gamma \in R_{k_i}$. Therefore (u_i, v_i) form a permissible system of parameters in R_{k_i} and $\nu(u_i) = 1/Q_i$. Notice also that $v_i = \alpha u_i + \beta w_i$, where $\alpha, \beta \in R_{k_i}$. Moreover, β is a unit in R_{k_i}, since the image of v_i is a regular parameter in $R_{k_i}/(u_i)R_{k_i}$. This implies that $v_i = \alpha x_i^t \delta_i + \beta z_i$ is also a regular parameter in $S_{k'_i}$ and (x_i, v_i) form a permissible system of parameters in $S_{k'_i}$. We set $y_i = v_i$ and observe that

$$\nu^*(x_i) = \frac{1}{t}\nu(u_i) = \frac{1}{tQ_i} \quad \text{and} \quad \nu^*(y_i) = \nu(v_i) = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}} = \frac{1}{Q_i} \cdot \frac{p'_{i+1}}{tq'_{i+1}}.$$

Therefore $\nu^*(y_i)/\nu^*(x_i) = p'_{i+1}/q'_{i+1}$. □

Remark 9.4. Corollary 9.3 shows that for every $i \geq 0$ the inclusion $R_{k_i} \subset S_{k'_i}$ has the stable form (9.1). However, the permissible parameters (x_i, y_i) of $S_{k'_i}$ are not admissible if $q'_{i+1} = 1$.

Notice also that by Lemma 8.5 and Corollary 9.3 for all $l \geq 1$ we have a commutative diagram

$$\begin{array}{ccccccccccccc} S_{s_{l-1}} & = & S_{k'_{i_{l-1}}} & \to & S_{k'_{i_{l-1}+1}} & \to & \dots & \to & S_{k'_{i_l-1}} & \to & S_{k'_{i_l}} & = & S_{s_l} \\ & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\ R_{r_{l-1}} & = & R_{k_{i_{l-1}}} & \to & R_{k_{i_{l-1}+1}} & \to & \dots & \to & R_{k_{i_l-1}} & \to & R_{k_{i_l}} & = & R_{r_l} \end{array} \tag{9.4}$$

where the sequences satisfy the conclusions of Corollary 9.3.

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