The model theory of tame valued fields
Preliminary version

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Abstract

A henselian valued field K is called a tame field if its separable-algebraic closure K^{sep} is a tame extension, that is, K^{sep} is equal to the ramification field of the normal extension $K^{\mathrm{sep}}|K$. Every algebraically maximal Kaplansky field is a tame field, but not conversely. We prove Ax–Kochen–Ershov Principles for tame fields. This leads to model completeness and completeness results relative to value group and residue field. As the maximal immediate extensions of tame fields will in general not be unique, the proofs have to use much deeper valuation theoretical results than those for other classes of valued fields which have already been shown to satisfy Ax–Kochen–Ershov Principles. The results of this paper have been applied to gain insight in the Zariski space of places of an algebraic function field, and in the model theory of large fields.

This is a preliminary version, with several proofs (in particular, the model theoretic nonsense) omitted. Apologies for any inconvenience due to lack of coherence in some places and for missing references.

1 Introduction

In this paper, we consider the model theory of valued fields. By (K, v) we mean a field K equipped with a valuation v. We denote the value group by vK, the residue field by Kv and the valuation ring by $\mathcal{O}_v$ or $\mathcal{O}_K$. For elements $a \in K$, the value is denoted by va, and the residue by av. We write a valuation in the classical additive (Krull) way, that is, the value group is an additively written ordered abelian group, the homomorphism property of v says that $vab = va+vb$, and the ultrametric triangle law says that $v(a+b) \geq \min\{va, vb\}$. Further, we have the rule $va = \infty \Leftrightarrow a = 0$. We take $\mathcal{L} = \{+, -, \cdot, 0, 1, \mathcal{O}\}$ to be the language of valued rings, where $\mathcal{O}$ is a binary relation symbol for valuation divisibility. That is, $\mathcal{O}(a, b)$ will be interpreted by $va \geq vb$, or equivalently, a/b lies in the valuation ring. We will write $\mathcal{O}(X)$ in the place of $\mathcal{O}(X, 1)$ (note that $\mathcal{O}(a, 1)$ says that $va \geq v1 = 0$, i.e., $a \in \mathcal{O}_v$).

When we talk of a valued field extension $(L|K, v)$ we mean that (L, v) is a valued field, $L|K$ a field extension, and K is endowed with the restriction of v.

A valued field is **henselian** if it satisfies Hensel's Lemma, or equivalently, if it admits a unique extension of the valuation to every algebraic extension field; see [Ri], [W], [E–P].

For (K,v) and (L,v) to be elementarily equivalent in the language of valued fields, it is necessary that vK and vL are elementarily equivalent in the language of ordered groups, and that Kv and Lv are elementarily equivalent in the language of rings (or fields). Our main concern in this paper is to find conditions under which these necessary conditions are also sufficient, i.e., the following **Ax–Kochen–Ershov Principle** (in short: AKE$^{\equiv}$ Principle) holds:

$$vK \equiv vL \;\wedge\; Kv \equiv Lv \quad\Longrightarrow\quad (K,v) \equiv (L,v)\;. \tag{1}$$

An AKE$^{\prec}$ Principle is the following analogue for elementary extensions:

$$(K,v) \subseteq (L,v) \;\wedge\; vK \prec vL \;\wedge\; Kv \prec Lv \quad\Longrightarrow\quad (K,v) \prec (L,v)\;. \tag{2}$$

Inspired by Robinson's Test, our basic approach will be to ask for criteria for a valued field to be existentially closed in a given extension field. Replacing $\equiv$ by $\prec_\exists$, we thus look for conditions which ensure that the following AKE$^{\exists}$ Principle holds:

$$(K,v) \subseteq (L,v) \;\wedge\; vK \prec_\exists vL \;\wedge\; Kv \prec_\exists Lv \quad\Longrightarrow\quad (K,v) \prec_\exists (L,v)\;. \tag{3}$$

The conditions

$$vK \prec_\exists vL \;\text{ and }\; Kv \prec_\exists Lv \tag{4}$$

will be called **side conditions**. It is an easy exercise in model theoretic algebra to show that these conditions imply that vL/vK is torsion free and that $Lv|Kv$ is regular, i.e., the algebraic closure of Kv is linearly disjoint from Lv over Kv, or equivalently, Kv is relatively algebraically closed in Lv and $Lv|Kv$ is separable (not necessarily algebraic); cf. Lemma 3.5.

A valued field for which (3) holds will be called an **AKE$^{\exists}$-field**. A class $\mathbf{C}$ of valued fields will be called **AKE$^{\equiv}$-class** (or **AKE$^{\prec}$-class**) if (1) (or (2), respectively) holds for all $(K,v),(L,v) \in \mathbf{C}$, and it will be called **AKE$^{\exists}$-class** if (3) holds for all $(K,v) \in \mathbf{C}$. We will also say that $\mathbf{C}$ is **relatively complete** if it is an AKE$^{\equiv}$-class, and that $\mathbf{C}$ is **relatively model complete** if it is an AKE$^{\prec}$-class. Here, "relatively" means "relative to the value groups and residue fields".

The following elementary classes of valued fields are known to satisfy all or some of the above AKE Principles:

a) Algebraically closed valued fields satisfy all three AKE Principles. They even admit quantifier elimination; this has been shown by Abraham Robinson, cf. [Ro].

b) Henselian fields of residue characteristic 0 satisfy all three AKE Principles. These facts have been (explicitly or implicitly) shown by Ax and Kochen [AK] and independently by Ershov. They admit quantifier elimination relative to their value group and residue field, cf. [D].

c) p-adically closed fields satisfy all three AKE Principles. Again, these fields were treated by Ax and Kochen [AK] and independently by Ershov.

d) $\wp$-adically closed fields (i.e., finite extensions of p-adically closed fields). See Prestel and Roquette [P–R].

e) Finitely ramified fields. This is a generalization of c) and d). These fields were treated by Ziegler [Zi] and independently by Ershov.

f) Algebraically maximal Kaplansky fields. Again, these fields were treated by Ziegler [Zi] and independently by Ershov.

An extension $(L|K,v)$ of valued fields is called **immediate** if the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto. A valued field is called **algebraically maximal** if it admits no proper immediate algebraic extensions; it is called **separable-algebraically maximal** if it does not admit proper immediate separable algebraic extensions.

Every valued field admits a maximal immediate algebraic extension and a maximal immediate extension. All of the above mentioned valued fields have the common property that these extensions are unique up to valuation preserving isomorphism. This has always been a nice tool in the proofs of the model theoretic results. However, as we will show in this paper, this uniqueness is not indispensible. One just has to work much harder.

We will show that tame valued fields (in short, "tame fields") form an AKE$^\exists$-class, and we will prove further model theoretic results for tame fields and separably tame fields. Take a henselian field (K,v), and let p denote the **characteristic exponent** of its residue field Kv, i.e., $p = \mathrm{char}\, Kv$ if this is positive, and $p = 1$ otherwise. An algebraic extension $(L|K,v)$ of a henselian field (K,v) is called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) The ramification index $(vE : vK)$ is prime to p,

(TE2) The residue field extension $Ev|Kv$ is separable,

(TE3) The extension $(E|K,v)$ is **defectless**, i.e., $[E:K] = (vE : vK)[Ev : Kv]$.

Remark 1.1 This notion of "tame extension" does not coincide with the notion of "tamely ramified extension" as defined in the book [En] of O. Endler (page 180). The latter definition requires (TE1) and (TE2), but not (TE3). Our tame extensions are the defectless tamely ramified extensions in the sense of Endler's book. In particular, in our terminology, proper immediate algebraic extensions of henselian fields are not called tame, because they cause a lot of problems in the model theory of valued fields.

A **tame field** is a henselian field for which all algebraic extensions are tame. Likewise, a **separably tame field** is a henselian field for which all separable-algebraic extensions are tame. All henselian fields of residue characteristic 0 and all algebraically maximal Kaplansky fields are tame fields.

In many applications (such as the proof of a Nullstellensatz), only existential sentences play a role. In these cases, it suffices to have an AKE$^\exists$ Principle at hand. There are situations where we cannot even expect more than this principle. In order to present one, we will need some definitions that will be fundamental for this paper.

Every finite extension $(L|K,v)$ of valued fields satisfies the **fundamental inequality** (cf. [En], [Ri], or [Z–S]):

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{5}$$

where $n = [L:K]$ is the degree of the extension, $v_1, \ldots, v_{\mathrm{g}}$ are the distinct extensions of v from K to L, $\mathrm{e}_i = (v_iL : vK)$ are the respective ramification indices and $\mathrm{f}_i = [Lv_i : Kv]$ are the respective inertia degrees. The extension is called **defectless** if equality holds in (5). Note that $\mathrm{g} = 1$ if (K,v) is henselian, so the definition given in axiom (TE3) is a special case of this definition.

A valued field (K,v) is called **defectless** (or **stable**) if each of its finite extensions is defectless, and **separably defectless** if each of its finite extensions is defectless. If $\mathrm{char}\, Kv = 0$, then (K,v) is defectless (this is a consequence of the "Lemma of Ostrowski", cf. (10).

Now let $(L|K,v)$ be any extension of valued fields. Assume that $L|K$ has finite transcendence degree. Then (by Corollary 2.2 below):

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{trdeg}\, Lv|Kv \,+\, \dim_{\mathbb{Q}} \mathbb{Q} \otimes vL/vK\;. \tag{6}$$

We will say that $(L|K,v)$ is **without transcendence defect** if equality holds in (6). In Section 4 we will prove:

Theorem 1.2 *Every extension without transcendence defect of a henselian defectless field satisfies the AKE$^{\exists}$ Principle.*

Note that it is not in general true that an extension of henselian defectless fields will satisfy the AKE$^{\prec}$ Principle. A counterexample is given in [K4].

For a similar notion, see our definition of "relatively subcomplete" in Section 5 below.

A valued field (K,v) has **equal characteristic** if $\mathrm{char}\, K = \mathrm{char}\, Kv$. The following is the main theorem of this paper:

Theorem 1.3 *The class of all tame fields is an AKE$^{\exists}$-class and an AKE$^{\prec}$-class. The class of all equal characteristic tame fields is an AKE$^{\equiv}$-class.*

As an immediate consequence of the foregoing Theorem, we get the following criterion for decidability:

Corollary 1.4 *Let (K,v) be a tame field of equal characteristic. Assume that the theories* $\mathrm{Th}(vK)$ *of its value group (as an ordered group) and* $\mathrm{Th}(Kv)$ *of its residue field (as a field) both admit recursive elementary axiomatizations. Then also the theory of (K,v) as a valued field admits a recursive elementary axiomatization, and it is decidable.*

Indeed, the axiomatization of $\mathrm{Th}(K,v)$ can be taken to consist of the axioms of tame fields of equal characteristic $\mathrm{char}\, K$, together with the translations of the axioms of $\mathrm{Th}(vK)$ and $\mathrm{Th}(Kv)$ to the language of valued fields (cf. Lemma 3.1).

Corollary 1.5 *Take Γ to be divisible or the p-divisible hull of $\mathbb{Z}$. Then the elementary theory of $\mathbb{F}_q((t^\Gamma))$, where $q = p^n$ for some $n \in \mathbb{N}$, is decidable.*

Here are our results for separably defectless and separably tame fields, which we will prove in Section 7:

Theorem 1.6 *a) Take an extension $(L|K,v)$ without transcendence defect of a henselian separably defectless field such that vK is cofinal in vL. Then the extension satisfies the AKE$^\exists$ Principle.*
b) Every separable extension of a separably tame field satisfies the AKE$^\exists$ Principle.

We will deduce our model theoretic results from two main theorems which we originally proved in [K1]. The first theorem is a generalization of the "Grauert–Remmert Stability Theorem". It deals with function fields $F|K$, i.e., F is a finitely generated field extension of K (for our purposes it is not necessary to ask that the transcendence degree is ≥ 1).

Theorem 1.7 *Let $(F|K,v)$ be a valued function field without transcendence defect. If (K,v) is a defectless field, then also (F,v) is a defectless field.*

In [K-K1] we use Theorem 1.7 to prove:

Theorem 1.8 *Take a defectless field (K,v) and a valued function field $(F|K,v)$ without transcendence defect. Assume that $Fv|Kv$ is a separable extension and vF/vK is torsion free. Then $(F|K,v)$ admits elimination of ramification in the following sense: there is a standard valuation transcendence basis $\mathcal{T} = \{x_1,\ldots,x_r,y_1,\ldots,y_s\}$ of $(F|K,v)$ such that*

a) $vF = vK \oplus \mathbb{Z}vx_1 \oplus \ldots \oplus \mathbb{Z}vx_r$,

b) $y_1v,\ldots,y_sv$ form a separating transcendence basis of $Fv|Kv$.

For each such transcendence basis $\mathcal{T}$ and every extension of v to the algebraic closure of F, $(F^h|K(\mathcal{T})^h,v)$ is unramified.

The second fundamental theorem is a structure theorem for immediate function fields over tame or separably tame fields.

Theorem 1.9 *Let $(F|K,v)$ an immediate function field of transcendence degree 1. Assume that (K,v) is a tame field, or that (K,v) is a separably tame field and $F|K$ is separable. Then*

$$\text{there is } x \in F \text{ such that } (F^h,v) = (K(x)^h,v)\,, \tag{7}$$

that is, $(F|K,v)$ is henselian generated.

For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless (in fact, every $x \in F \setminus K$ will then do the job). In contrast to this, the case of positive residue characteristic requires a much deeper structure theory of immediate algebraic extensions of henselian fields, in order to find suitable elements x.

2 Algebraic preliminaries

2.1 Valuation theoretical preliminaries

For basic facts from valuation theory, see [En], [Ri], [W], [E–P], [Z–S], [K2].

We will denote the algebraic closure of a field K by $\tilde{K}$. Whenever we have a valuation v on K, we will automatically fix an extension of v to the algebraic closure $\tilde{K}$ of K. It does not play a role which extension we choose, except if v is also given on an extension field L of K; in this case, we choose the extension to $\tilde{K}$ to be the restriction of the extension to $\tilde{L}$. We say that v is **trivial** on K if $vK = \{0\}$. If v is given on some extension field L of K and is trivial on K, then there is some valuation v' of L which is equivalent to v (i.e., they have the same valuation ring on L) and whose restriction to K is the identity.

For the easy proof of the following lemma, see [B], chapter VI, §10.3, Theorem 1.

Lemma 2.1 *Let $(L|K,v)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values vx_i, $i \in I$, are rationally independent over vK, and the residues y_jv, $j \in J$, are algebraically independent over Kv. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.*

Moreover, if we write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i\in I, j\in J] \tag{8}$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf \;=\; \min_k \, v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k \, vc_k + \sum_{i\in I} \mu_{k,i} v x_i \,. \tag{9}$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i\in I, j\in J) &= vK \oplus \bigoplus_{i\in I} \mathbb{Z} v x_i \\ K(x_i, y_j \mid i\in I, j\in J)v &= Kv\,(y_jv \mid j\in J)\,. \end{aligned}$$

The valuation v on $K(x_i, y_j \mid i\in I, j\in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues y_jv.

Corollary 2.2 *Take a valued function field $(F|K,v)$ without transcendence defect and set $r = \dim_{\mathbb{Q}} \mathbb{Q} \otimes vF/vK$ and $s = \mathrm{trdeg}\, Fv|Kv$. Choose elements $x_1, \ldots, x_r$, $y_1, \ldots, y_s \in F$ such that the values $vx_1, \ldots, vx_r$ are rationally independent over vK and the residues $y_1v, \ldots, y_sv$ are algebraically independent over Kv. Then $\mathcal{T} = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ is a standard valuation transcendence basis $(F|K,v)$ Moreover, vF/vK and the extension $Fv|Kv$ are finitely generated.*

Proof: By the foregoing theorem, the elements $x_1,\ldots,x_r,y_1,\ldots,y_s$ are algebraically independent over K. Since $\operatorname{trdeg} F|K = r+s$ by assumption, these elements form a basis and hence a standard valuation transcendence basis of $(F|K,v)$.

It follows that the extension $F|K(\mathcal{T})$ is finite. By the fundamental inequality (5), this yields that $vF/vK(\mathcal{T})$ and $Fv|K(\mathcal{T})v$ are finite. Since already $vK(\mathcal{T})/vK$ and $K(\mathcal{T})v|Kv$ are finitely generated by the foregoing lemma, it follows that also vF/vK and $Fv|Kv$ are finitely generated. □

The **henselization** (K^h,v) of a valued field (K,v) (in its separable-algebraic closure (K^{sep},v)) is the minimal henselian extension of (K,v), in the following sense: if (L,v') is a henselian extension field of (K,v), then there is a unique embedding of (K^h,v) in (L,v'). This is the **universal property of the henselization**. We note that every algebraic extension of a henselian field is again henselian. In particular, since the absolute inertia field of an arbitrary valued field contains its henselization, it is henselian.

We also note:

Lemma 2.3 *If K is an arbitrary field and v is a valuation on K^{sep}, then vK^{sep} is the divisible hull of vK, and $(Kv)^{\mathrm{sep}} \subseteq K^{\mathrm{sep}}v$. If in addition v is non-trivial on K, then $K^{\mathrm{sep}}v$ is the algebraic closure of Kv.*

We leave the easy proof to the reader.

2.2 The defect

Assume that $(L|K,v)$ is a finite extension and the extension of v from K to L is unique (which is always the case when (K,v) is henselian). Then the Lemma of Ostrowski (cf. [En], [Ri], [K2]) says that

$$[L:K] \;=\; (vL:vK)\cdot[Lv:Kv]\cdot p^{\nu} \quad \text{with } \nu\geq 0 \tag{10}$$

where p is the **characteristic exponent** of Kv, that is, $p=\operatorname{char} Kv$ if this is positive, and $p=1$ otherwise. The factor

$$\mathrm{d}(L|K,v) \;:=\; p^{\nu} \;=\; \frac{[L:K]}{(vL:vK)[Lv:Kv]}$$

is called the **defect** of the extension $(L|K,v)$. If $\nu>0$, then we talk of a **non-trivial** defect. If $[L:K]=p$ then $(L|K,v)$ has non-trivial defect if and only if it is immediate, i.e., $(vL:vK)=1$ and $[Lv:Kv]=1$. If $\mathrm{d}(L|K,v)=1$, then we call $(L|K,v)$ a **defectless extension**. Note that $(L|K,v)$ is always defectless if $\operatorname{char} Kv=0$. Therefore,

Corollary 2.4 *Every valued field (K,v) with $\operatorname{char} Kv=0$ is a defectless field.*

The following lemma shows that the defect is multiplicative. This is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. We leave the straightforward proof to the reader.

Lemma 2.5 *Fix an extension of a valuation v from K to its algebraic closure. If $L|K$ and $M|L$ finite extensions and the extension of v from K to M is unique, then*

$$\mathrm{d}(M|K,v) = \mathrm{d}(M|L,v)\cdot \mathrm{d}(L|K,v) \tag{11}$$

In particular, $(M|K,v)$ is defectless if and only if $(M|L,v)$ and $(L|K,v)$ are defectless.

A valued field (K,v) is called **separably defectless** if equality holds in (5) for every finite separable extension, and it is called **inseparably defectless** if equality holds in (5) for every finite purely in separable extension $L|K$.

Theorem 2.6 *Take a valued field (K,v) and fix an extension of v to $\tilde{K}$. Then (K,v) is defectless if and only if its henselization $(K,v)^h$ in $(\tilde{K},v)$ is defectless. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".*

Proof: For "separably defectless", our assertion follows directly from [En], Theorem (18.2). The proof of that theorem can easily be adapted to prove the assertion for "inseparably defectless" and "defectless". See [K2] for more details. □

Since a henselian field has a unique extension of the valuation to every algebraic extension field, we obtain:

Corollary 2.7 *A valued field (K,v) is defectless if and only if $\mathrm{d}(L|K^h,v)=1$ for every finite extension $L|K^h$.*

Using this corollary together with Lemma 2.5, one easily shows:

Corollary 2.8 *Every finite extension of a defectless field is again a defectless field.*

Lemma 2.9 *Let (K,v) be a valued field with $v = w\circ\overline{w}$. If (K,w) and $(Kw,\overline{w})$ are defectless fields, then so is (K,v).*

Proof: Take a finite extension $L|K$; we wish to show that equality holds in (5). Let $w_1,\ldots,w_{\mathrm{g}_w}$ be all extensions of w from K to L, and set $\mathrm{e}_i^w=(w_iL:wK)$ and $\mathrm{f}_i^w=[Lw_i:Kw]$ for $1\leq i\leq \mathrm{g}_w$. Further, for $1\leq i\leq \mathrm{g}_w$, let $\overline{w}_{i1},\ldots,\overline{w}_{i\mathrm{g}_i}$ be all extensions of $\overline{w}$ from Kw to Lw_i, and set $\mathrm{e}_{ij}=(\overline{w}_{ij}(Lw_i):\overline{w}(Kw))$ and $\mathrm{f}_{ij}=[(Lw_i)\overline{w}_{ij}:(Kw)\overline{w}]=[L(w_i\circ\overline{w}_{ij}):Kv]$ for $1\leq j\leq \mathrm{g}_i$. Since (K,w) is a defectless field, we have $[L:K]=\sum_{i=1}^{\mathrm{g}_w}\mathrm{e}_i^w\mathrm{f}_i^w$. Since $(Kw,\overline{w})$ is a defectless field, we have $\mathrm{f}_i^w=[Lw_i:Kw]=\sum_{j=1}^{\mathrm{g}_i}\mathrm{e}_{ij}\mathrm{f}_{ij}$. Using that $\mathrm{e}_i^w\mathrm{e}_{ij}=((w_i\circ\overline{w}_{ij})L:vK)$, we obtain:

$$[L:K]\;=\;\sum_{i=1}^{\mathrm{g}_w}\mathrm{e}_i^w\sum_{j=1}^{\mathrm{g}_i}\mathrm{e}_{ij}\mathrm{f}_{ij}\;=\;\sum_{i=1}^{\mathrm{g}_w}\sum_{j=1}^{\mathrm{g}_i}((w_i\circ\overline{w}_{ij})L:vK)[L(w_i\circ\overline{w}_{ij}):Kv]\,.$$

As the valuations $w_i\circ\overline{w}_{ij}$, $1\leq i\leq \mathrm{g}_w$, $1\leq j\leq \mathrm{g}_i$, are distinct extensions of v from K to L, the fundamental inequality implies that they are in fact all extensions, and we have proved that equality holds in (5). □

We will denote by K^r the ramification field of the normal extension $K^{\mathrm{sep}}|K$. In [K2] we have proved the following:

Proposition 2.10 *Let (K,v) be a henselian field and N an arbitrary algebraic extension of K within K^r. If $L|K$ is a finite extension, then*

$$\mathrm{d}(L|K,v) \;=\; \mathrm{d}(L.N|N,v)\;.$$

Hence, (K,v) is a defectless field if and only if (N,v) is a defectless field. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".

Lemma 2.11 *Let v be a valuation of $F|K$ and suppose that E is a subfield of F on which v is trivial. Let (F^i,v) denote the absolute inertia field of (F,v). Then $E^{\mathrm{sep}}\subset F^i$. Further, if $Fv|Ev$ is algebraic, then $(F.E^{\mathrm{sep}})v=(Fv)^{\mathrm{sep}}$.*

Proof: By assumption, v induces an embedding of E in Fv. Further, we know by ramification theory ([En], [K2]) that F^iv is separable-algebraically closed. Thus, $(Ev)^{\mathrm{sep}}\subset F^iv$. Using Hensel's Lemma, one shows that the inverse of the isomorphism $E\ni a\mapsto av\in Ev$ can be extended from Ev to an embedding of $(Ev)^{\mathrm{sep}}$ in F^i. Its image is separable-algebraically closed and contains E. Hence, $E^{\mathrm{sep}}\subset F^i$. Further, $(F.E^{\mathrm{sep}})v$ contains $E^{\mathrm{sep}}v$, which by Lemma 2.3 contains $(Ev)^{\mathrm{sep}}$. As $F.E^{\mathrm{sep}}|F$ is algebraic, so is $(F.E^{\mathrm{sep}})v|Fv$. Therefore, if $Fv|Ev$ is algebraic, then $(F.E^{\mathrm{sep}})v$ is algebraic over $(Ev)^{\mathrm{sep}}$ and hence separable-algebraically closed. Since $(F.E^{\mathrm{sep}})v\subset F^iv=(Fv)^{\mathrm{sep}}$, it follows that $(F.E^{\mathrm{sep}})v=(Fv)^{\mathrm{sep}}$. □

2.3 Algebraically maximal and separable-algebraically maximal fields

All algebraically maximal and all separable-algebraically maximal fields are henselian because the henselization is an immediate separable-algebraic extension and therefore these fields must coincide with their henselization.

We will assume the reader to be familiar with the theory of pseudo Cauchy sequences as developed in [Ka]. See this paper for the proofs of the following two theorems.

Theorem 2.12 *A valued field (K,v) is algebraically maximal if and only if every pseudo Cauchy sequence in (K,v) without a pseudo limit in K is of transcendental type, i.e., it eventually fixes the value of every polynomial in $K[X]$.*

Theorem 2.13 *Let (K,v) be an algebraically maximal field and $(K(x)|K,v)$ and $(K(y)|K,v)$ two non-trivial immediate extensions. If x and y are limits of the same pseudo Cauchy sequence in (K,v) without a limit in K, then $x\mapsto y$ induces a valuation preserving isomorphism from $K(x)$ to $K(y)$.*

We will need the following characterizations of algebraically maximal and separable-algebraically maximal fields. For the proofs, see [K9] and [K2].

Theorem 2.14 *The valued field (K,v) is algebraically maximal if and only if it is henselian and for every polynomial $f\in K[X]$,*

$$\exists x\in K\,\forall y\in K:\; vf(x)\geq vf(y)\;. \tag{12}$$

Similarly, (K,v) is separable-algebraically maximal if and only if (12) holds for every separable polynomial $f\in K[X]$.

2.4 Algebraic properties of tame fields

An algebraic extension of a henselian field is called **purely wild** if it is linearly disjoint from every tame extension. We will call (K,v) a **purely wild field** if $(\tilde{K}|K,v)$ is a purely wild extension. For example, for every henselian field (K,v), its ramification field $(K,v)^r$ is trivially a purely wild field.

The first part of the lemma follows from the fact that the absolute ramification field K^r is the unique maximal tame extension of a henselian field K. The second part follows from the fact that for every algebraic extension $L|K$, $L^r=L.K^r$. These facts follow from general ramification theory; see [En], [K2].

Lemma 2.15 *a) Let (K,v) be a henselian field. Then (K,v) is a tame field if and only if $K^r=\tilde{K}$. Similarly, (K,v) is a separably tame field if and only if $K^r=K^{\rm sep}$. Further, (K,v) is a purely wild field if and only if $K^r=K$.*

b) Every algebraic extension of a tame (resp. separably tame, resp. purely wild) field is again a tame (resp. separably tame, resp. purely wild) field.

If (K,v) is a henselian field of residue characteristic 0, then every algebraic extension $(L|K,v)$ is tame, as we have seen in the last section. So we note:

Lemma 2.16 *Every algebraic extension of henselian fields of residue characteristic 0 is a tame extension. Every henselian field of residue characteristic 0 is a tame field.*

From the definition and the fact that every tame extension is defectless and separable, we obtain:

Lemma 2.17 *Every tame field is henselian defectless and perfect.*

In general, infinite extensions of defectless fields need not be defectless fields. For example, $\mathbb{F}_p(t)^h$ is a defectless field by Theorem 1.7 and Theorem 2.6, but the perfect hull of $\mathbb{F}_p(t)^h$ is a henselian field admitting an immediate extension generated by a root of the polynomial $X^p-X-\frac{1}{t}$. But from Lemma 2.15 and the foregoing lemma, we can deduce:

Corollary 2.18 *Every algebraic extension of a tame field is a tame field and hence a defectless field.*

The following theorem was proved by M. Pank; cf. [K–P–R].

Theorem 2.19 *Let (K,v) be a henselian field with residue characteristic $p>0$. There exist algebraic field complements W_s of K^{sep} over K, i.e. $K^r.W_s = K^{\mathrm{sep}}$ and W_s is linearly disjoint from K^{sep} over K. The perfect hull $W = W_s^{1/p^\infty}$ is an algebraic field complement of K^r over K, i.e. $K^r.W = \tilde{K}$ and W is linearly disjoint from K^r over K. The valued complements (W_s,v) can be characterized as the maximal separable algebraic purely wild extensions of (K,v), and the (W,v) are the maximal algebraic purely wild extensions of (K,v).*

Using Theorem 2.19, we give some characterizations for tame fields.

Lemma 2.20 *The following assertions are equivalent:*
1) (K,v) is tame,
2) Every algebraic purely wild extension $(L|K,v)$ is trivial,
3) (K,v) is algebraically maximal and closed under purely wild extensions by p-th roots,
4) (K,v) is algebraically maximal, vK is p-divisible and Kv is perfect.

Proof: Let (K,v) be a tame field, i.e., $K^r = \tilde{K}$. Then by definition, every algebraic purely wild extension of (K,v) must be trivial. This proves 1)⇒2).

Suppose that (K,v) has no algebraic purely wild extension. Then in particular, it has no purely wild extension by p-th roots. Since every immediate algebraic extension of a henselian field is purely wild by definition, we also obtain that (K,v) admits no proper immediate algebraic extension, i.e., (K,v) is algebraically maximal. This proves 2)⇒3).

Assume now that (K,v) is an algebraically maximal field closed under purely wild extensions by p-th roots. Let a be an arbitrary element of K. Assume that va is not divisible by p in vK; then the extension $K(b)|K$ generated by an element $b\in\tilde{K}$ with $b^p = a$ satisfies $(vK(b):vK) = p = [K(b):K]$ and thus admits a unique extension of v. With this extension it is purely wild, contrary to our assumption on (K,v). Assume that $va = 0$ and that av has no p-th root in Kv; then the extension $K(b)|K$ generated as above satisfies $[K(b)v:Kv] = p = [K(b):K]$ and is again purely wild, contrary to our assumption. By this, we have shown that vK is p-divisible and Kv is perfect. This proves 3)⇒4).

Suppose that (K,v) is an algebraically maximal (and thus henselian) field such that vK is p-divisible and Kv is perfect. Choose a maximal purely wild extension (W,v) in accordance to Theorem 2.19. Our condition on the value group and the residue field yields that $(W|K,v)$ is immediate. But since (K,v) is assumed to be algebraically maximal, this extension must be trivial. This shows that $\tilde{K} = K^r$, i.e., (K,v) is a tame field. This proves 4)⇒1). □

If K has characteristic $p>0$, then every extension by p-th roots is purely inseparable and thus purely wild. So the lemma yields:

Corollary 2.21 *A valued field (K,v) of characteristic $p>0$ is tame if and only if it is algebraically maximal and perfect. Consequently, if (K,v) is an arbitrary valued field of characteristic $p>0$, then every maximal immediate algebraic extension (W,v) of $(K^{1/p^\infty},v)$ is a tame field.*

For perfect valued fields of positive characteristic, the properties "algebraically maximal" and "henselian defectless" are equivalent.

The next corollary shows how to construct tame fields with suitable prescribed value group and residue field:

Corollary 2.22 *Let p be a prime number, Γ a p-divisible ordered abelian group and k a perfect field of characteristic p. Then there exists a tame field K of characteristic p having Γ as its value group and k as its residue field such that $K|\mathbb{F}_p$ admits a valuation transcendence basis and the cardinality of K is equal to the maximum of the cardinalities of Γ and k.*

Proof: According to Theorem 2.14. of [K7], there is a valued field (K_0, v) of characteristic p with value group Γ and residue field k, and admitting a valuation transcendence basis over its prime field. Now take (K, v) to be a maximal immediate algebraic extension of (K_0, v). Then (K, v) is algebraically maximal, and Lemma 2.20 shows that it is a tame field. Since it is an algebraic extension of (K_0, v), it still admits a valuation transcendence basis over its prime field. Hence, it follows that $|K| = \max\{|\Gamma|, |k|\}$. (If v is nontrivial, then K is infinite. If v is trivial, then $\Gamma = \{0\}$ and $K = k$.) □

Now we will prove an important lemma on tame fields that we will need in several instances.

Lemma 2.23 *Let (L, v) be a tame field and $K \subset L$ a relatively algebraically closed subfield. If in addition $Lv|Kv$ is an algebraic extension, then K is also a tame field and moreover, vK is pure in vL and $Kv = Lv$.*

Proof: The following short and elegant version of the proof was given by Florian Pop. Since (L, v) is tame, it is henselian and perfect. Since K is relatively algebraically closed in L, it is henselian and perfect too. Assume that $(K_1|K, v)$ is a finite purely wild extension; in view of Lemma 2.20, we have to show that it is trivial. The degree $[K_1 : K]$ is a power of p, say p^m. Since K is perfect, $L|K$ and $K_1|K$ are separable extensions. Since K is relatively algebraically closed in L, we know that L and K_1 are linearly disjoint over K. Thus, K_1 is relatively algebraically closed in $K_1.L$, and

$$[K_1.L : L] = [K_1 : K] = p^m .$$

Since L is assumed to be a tame field, the extension $(K_1.L|L, v)$ must be tame. This implies that

$$(K_1.L)v \mid Lv$$

is a separable extension of degree p^m. On the other hand, $(K_1.L)v \mid K_1v$ is an algebraic extension since by hypothesis, $Lv \mid Kv$ and thus also $(K_1.L)v \mid Kv$ are algebraic extensions. Furthermore, $(K_1.L, v)$ being a henselian field and K_1 being relatively algebraically closed in $K_1.L$, Hensel's Lemma shows that

$$(K_1.L)v \mid K_1v$$

must be purely inseparable. This yields that

$$\begin{aligned} p^m &= [(K_1.L)v : Lv]_{\rm sep} \le [(K_1.L)v : Kv]_{\rm sep} = [(K_1.L)v : K_1v]_{\rm sep} \cdot [K_1v : Kv]_{\rm sep} \\ &= [K_1v : Kv]_{\rm sep} \le [K_1v : Kv] \le [K_1 : K] = p^m \;, \end{aligned}$$

showing that equality holds everywhere, which implies that

$$K_1v \,|\, Kv$$

is separable of degree p^m. Since $K_1|K$ was assumed to be purely wild, we have $p^m = 1$ and the extension $K_1|K$ is trivial.

We have now shown that K is a tame field; hence by Lemma 2.20, vK is p-divisible and $\overline{K}$ is perfect. Since $\overline{L}|\overline{K}$ is assumed to be algebraic, one can use Hensel's Lemma to show that $Lv = Kv$ and that the torsion subgroup of vL/vK is a p-group. But as vK is p-divisible, vL/vK has no p-torsion, showing that vL/vK has no torsion at all. □

A similar fact holds for separably tame fields, as stated in Lemma 2.31 below. The following corollaries will show some nice properties of the class of tame fields. They also possess generalizations to separably tame fields, see Corollary 2.32 below.

Corollary 2.24 *For every valued function field F with given transcendence basis $\mathcal{T}$ over a tame field K, there exists a tame subfield K_0 of K of finite rank with $K_0v = Kv$ and vK_0 pure in vK, and furthermore a function field F_0 with transcendence basis $\mathcal{T}$ over K_0 such that*

$$F = K.F_0 \tag{13}$$

and

$$[F_0 : K_0(\mathcal{T})] = [F : K(\mathcal{T})] \;. \tag{14}$$

Proof: Let $F = K(\mathcal{T})(a_1, \ldots, a_n)$. There exists a finitely generated subfield K_1 of K such that $a_1, \ldots, a_n$ are algebraic over $K(\mathcal{T})$ and $[F : K(\mathcal{T})] = [K_1(\mathcal{T})(a_1, \ldots, a_n) : K_1(\mathcal{T})]$. This will also hold for every extension field of K_1 in K. As a finitely generated field, (K_1, v) has finite rank. Now let y_j, $j \in J$, be a system of elements in K such that the residues y_jv, $j \in J$, form a transcendence basis of Kv over K_1v. According to Lemma 2.1, the field $K_1(y_j|j \in J)$ has residue field $K_1v(y_jv|j \in J)$ and the same value group as K_1, hence it is again a field of finite rank. Let K_0 be the relative closure of this field within K. Since by construction, $Kv|K_1v(y_jv|j \in J)$ and thus also $Kv|K_0v$ are algebraic, we can infer from the preceding lemma that K_0 is a tame field with $K_0v = Kv$ and vK_0 pure in vK. As an algebraic extension of a field of finite rank it is itself of finite rank. Finally, the function field $F_0 = K_0(\mathcal{T})(a_1, \ldots, a_n)$ has transcendence basis $\mathcal{T}$ over K_0 and satisfies assertions (13) and (14). □

Corollary 2.25 *For every extension $(L|K, v)$ with (L, v) a tame field, there exists a tame intermediate field L_0 such that the extension $(L_0|K, v)$ admits a valuation transcendence basis and the extension $(L|L_0, v)$ is immediate.*

Proof: Take $\mathcal{T}$ to be a maximal algebraically valuation independent set in $(L|K,v)$. With this choice, $vL/vK(\mathcal{T})$ is a torsion group and $Lv|K(\mathcal{T})v$ is algebraic. Let L_0 be the relative algebraic closure of $K(\mathcal{T})$ within L. Then by Lemma 2.23, we have that (L_0,v) is a tame field, that $Lv = L_0v$ and that vL_0 is pure in vL and thus $vL_0 = vL$. This shows that the extension $(L|L_0,v)$ is immediate. On the other hand, $\mathcal{T}$ is a valuation transcendence basis of $(L_0|K,v)$ by construction. □

2.5 Algebraic properties of separably tame fields

Note that "henselian separably defectless" implies "separable-algebraically maximal".

Since every finite separable algebraic extension of a separably tame field is tame and thus defectless, a separably tame field is always henselian separably defectless. The converse is not true; it needs additional assumptions on the value group and the residue field. Under the assumptions that we are going to use frequently, the converse will even hold for "separable-algebraically maximal" in the place of "henselian separably defectless". Before proving this, we need a lemma which makes essential use of Theorem 2.19.

Lemma 2.26 *A henselian field (K,v) is defectless if and only if every finite purely wild extension of (K,v) is defectless. Similarly, (K,v) is separably defectless if and only if every finite separable purely wild extension of (K,v) is defectless.*

Proof: By Theorem 2.19, there exists a field complement W of K^r over K in K^{sep}, and W^{1/p^∞} is a field complement of K^r in $\tilde{K}$. Consequently, given any finite extension (resp. finite separable extension) $(L|K,v)$, there is a finite subextension $N|K$ of $K^r|K$ and a finite (resp. finite separable) subextension $W_0|K$ of $W|K$ such that $L \subset N.W_0$. If $(N.W_0|K,v)$ is defectless, then so is $(L|K,v)$; hence (K,v) is defectless (resp. separably defectless), if and only if every such extension $(N.W_0|K,v)$ is defectless. Since $(N|K,v)$ is a tame extension Lemma 2.10 shows that

$$d(N.W_0|N,v) = d(W_0|K,v)\ .$$

Hence, $(L|K,v)$ is defectless if and only if $(W_0|K,v)$ is defectless. This yields our assertion. □

Lemma 2.27 *The following assertions are equivalent:*
1) (K,v) is separably tame,
2) Every separable algebraic purely wild extension $(L|K,v)$ is trivial,
3) (K,v) is separable-algebraically maximal and closed under purely wild Artin-Schreier extensions,
4) (K,v) is separable-algebraically maximal, vK is p-divisible and Kv is perfect.

Proof: Let (K, v) be a separably tame field, i.e., $K^r = K^{\rm sep}$. Then by definition, every separable algebraic purely wild extension of (K, v) must be trivial. This proves 1)⇒2).

Now suppose that every separable algebraic purely wild extension of (K, v) is trivial. Then in particular, (K, v) admits no purely wild Artin-Schreier extensions (because they are separable). Furthermore, (K, v) admits no proper separable algebraic immediate extension since they are also purely wild. Consequently, (K, v) is separable-algebraically maximal. This proves 2)⇒3).

If (K, v) is closed under purely wild Artin-Schreier extensions, then vK is p-divisible and Kv is perfect (cf. Corollary 2.17 of [K7]). This proves 3)⇒4).

Suppose that (K, v) is a separable-algebraically maximal field such that vK is p-divisible and Kv is perfect. Then (K, v) is henselian. Choose a maximal separable algebraic purely wild extension (W_s, v) in accordance to Theorem 2.19. Our condition on the value group and the residue field yields that $(W_s|K, v)$ is immediate. But since (K, v) is assumed to be separable-algebraically maximal, this extension must be trivial. This shows that $K^{\rm sep} = K^r$, i.e., (K, v) is a separably tame field. This proves 4)⇒1). □

Suppose that (K, v) separably tame. Choose (W_s, v) according to Theorem 2.19. Then by condition 2) of the lemma, the extension $(W_s|K, v)$ must be trivial. This yields that $(K^{1/p^\infty}, v)$ is the unique maximal algebraic purely wild extension of $((K, v)$. Further, (K, v) also satisfies condition 3) of the lemma. From Corollary 4.6 of [K9] it follows that (K, v) is dense in $(K^{1/p^\infty}, v)$, i.e., K^{1/p^∞} lies in the completion of (K, v). This proves:

Corollary 2.28 *If (K, v) is separably tame, then the perfect hull K^{1/p^∞} of K is the unique maximal algebraic purely wild extension of $((K, v)$ and lies in the completion of (K, v). That is, every immediate algebraic extension of a separably tame field (K, v) is purely inseparable and included in the completion of (K, v), and every algebraic approximation type over (K, v) has distance ∞.*

Lemma 2.29 *(K, v) is a separably tame field if and only if $(K^{1/p^\infty}, v)$ is a tame field. Consequently, if $(K^{1/p^\infty}, v)$ is a tame field, then the extension (K, v) is dense in $(K^{1/p^\infty}, v)$.*

Proof: Suppose that (K, v) is a separably tame field. Then $(K^{1/p^\infty}, v)$ admits no purely wild algebraic extensions since otherwise, it would contain a proper separable algebraic purely wild subextension. Hence by Lemma 2.20, $(K^{1/p^\infty}, v)$ is a tame field.

For the converse, suppose that $(K^{1/p^\infty}, v)$ is a tame field. Observe that the extension $(K^{1/p^\infty}|K, v)$ is purely wild and contained in every maximal purely wild algebraic extension of (K, v). Consequently, if $(K^{1/p^\infty}, v)$ admits no purely wild algebraic extension at all, then $(K^{1/p^\infty}, v)$ is the unique maximal purely wild extension of (K, v). Then in view of Theorem 2.19, K^{1/p^∞} must be a field complement for K^r over K in $\tilde{K}$. This yields that $K^r = K^{\rm sep}$, i.e., $(K^{\rm sep}|K, v)$ is a tame extension, showing that (K, v) is a separably tame field. By the foregoing corollary, it follows that (K, v) is dense in $(K^{1/p^\infty}, v)$. □

The following lemma describes the behaviour of separably tame fields under a decomposition of their place.

Lemma 2.30 *Let (K,v) be a separably tame field and let P be the place associated with v. Assume $P = P_1P_2P_3$ where P_1 is a coarsening of P and P_2 is nontrivial. (P_3 may be trivial.) Then (KP_1, P_2) is a separably tame field. If also P_1 is nontrivial, then (KP_1, P_2) is a tame field.*

Proof: By Lemma 2.20, vK is p-divisible. The same is then true for $v_{P_2}KP_1$. We wish to show that the residue field KP_1P_2 is perfect. Indeed, assume that this were not the case. Then there is an Artin-Schreier extension of (K, P_1P_2) which adjoins a p-th root to the residue field KP_1P_2. Since already this residue field extension is purely inseparable, the induced extension of the residue field $Kv = KP_1P_2P_3$ can not be separable of degree p. This shows that the constructed Artin-Schreier extension is a separable algebraic purely wild extension of (K,v), contrary to our assumption on (K,v).

By Lemma 2.27, (K,P) is separable-algebraically maximal. This yields that the same is true for (K,P_1P_2). If P_1 is trivial (hence w.l.o.g. equal to the identity map), then $(KP_1,P_2) = (K,P_1P_2)$ is separable-algebraically maximal, and it follows from Lemma 2.27 that (KP_1,P_2) is a separably tame field. If P_1 is nontrivial, then it can be shown that (KP_1,P_2) is an algebraically maximal field, and it follows from Lemma 2.20 that (KP_1,P_2) is a tame field. □

The following is an analogue of Lemma 2.23.

Lemma 2.31 *Let (K,v) be a separably tame field and $k \subset K$ a relatively algebraically closed subfield of K. If the residue field extension $Kv|kv$ is algebraic, then (k,v) is also a separably tame field.*

Proof: Since k is relatively algebraically closed in K, it follows that also k^{1/p^∞} is relatively algebraically closed in K^{1/p^∞}. Since (K,v) is a separably tame field, $(K^{1/p^\infty},v)$ is a tame field by Lemma 2.29. From this lemma we also know that $Kv == K^{1/p^\infty}v$ and $vK = vK^{1/p^\infty}$. Our assumption on $Kv\,|\,kv$ yields that the extension $K^{1/p^\infty}v\,|\,k^{1/p^\infty}v$ is algebraic. From Lemma 2.23 we can now infer that $(k^{1/p^\infty},v)$ is a tame field with $k^{1/p^\infty}v = K^{1/p^\infty}v = Kv$ and vk^{1/p^∞} pure in $vK^{1/p^\infty} = vK$. Again by Lemma 2.29, (k,v) is thus a separably tame field with $kv = k^{1/p^\infty}v = Kv$ and $vk = vk^{1/p^\infty}$ pure in vK. □

Corollary 2.32 *Corollary 2.24 also holds for separably tame fields in the place of tame fields. More precisely, if $F|K$ is a separable extension, then F_0 and K_0 can be chosen such that $F_0|K_0$ (and thus also $F_0^h|K_0$) is a separable extension. Moreover, if vK is cofinal in vF then it can also be assumed that vK_0 is cofinal in vF_0.*

Proof: Since the proof of Corollary 2.24 only involves Lemma 2.23, it can be adapted by use of Lemma 2.31. The first additional assertion can be shown using the fact that the finitely generated separable extension $F|K$ is separably generated. The second additional assertion is seen as follows. If vF admits a biggest proper convex subgroup, then let K_0 contain a nonzero element whose value does not lie in this subgroup. If vF and thus also

vK does not admit a biggest proper convex subgroup, then first choose F_0 and K_0 as in the (generalized) proof of Lemma 2.24; since F_0 has finite rank, there exists some element in K whose value does not lie in the convex hull of vF_0 in vF, and adding this element to K_0 and F_0 will make vK_0 cofinal in vF_0. □

With the same proof as for Corollary 2.25, but using Lemma 2.31 in the place of Lemma 2.23, one shows:

Corollary 2.33 *Corollary 2.25 also holds for separably tame fields in the place of tame fields.*

3 Model theoretic preliminaries

3.1 Axioms for valued fields

A valuation v on a field K can be given in several ways. We can take the **valuation divisibility relation** and formalize it as a binary predicate R_v which in every valued field is to be interpreted as

$$R_v(x,y) \iff vx \geq vy\;.$$

But we can also take the valuation ring and formalize it as a predicate $\mathcal{O}$ which in every valued field is to be interpreted as

$$\mathcal{O}(x) \iff x \in \mathcal{O}\;.$$

This predicate can be defined from the valuation divisibility relation by

$$\mathcal{O}(x) \;\leftrightarrow\; R_v(x,1)\;.$$

If we are working in the language of fields (what we usually do), then the valuation divisibility relation can be defined from the predicate $\mathcal{O}$ by

$$R_v(x,y) \;\leftrightarrow\; (y \neq 0 \,\wedge\, \mathcal{O}(xy^{-1})) \,\vee\, x = 0\;,$$

whereas in general, it can not be defined using $\mathcal{O}$ and the language of rings without the use of quantifiers like in

$$R_v(x,y) \;\leftrightarrow\; (\exists z\, yz = 1 \,\wedge\, \mathcal{O}(xz)) \,\vee\, x = 0\;.$$

This fact is only of importance for questions of quantifier elimination, and only if one has decided to work in the language of rings. Note that two fields are equivalent in the language of rings if and only if they are equivalent in the language of fields. A similar assertion holds for valued fields in the respective languages, and it also holds for the notions “elementary extension” and “existentially closed in” in the place of “equivalent”.

We prefer to write "$vx \geq vy$" in the place of "$R_v(x,y)$". For convenience, we define the following relations:

$$\begin{aligned} vx > vy &\leftrightarrow vx \geq vy \wedge \neg(vy \geq vx) \\ vx = vy &\leftrightarrow vx \geq vy \wedge vy \geq vx \,. \end{aligned}$$

The definitions for the reversed relations $vx \leq vy$ and $vx < vy$ are left to the reader.

We will work in the **language of valued fields** $\mathcal{L}_{\mathrm{VF}}$ which is the language $\mathcal{L}_{\mathrm{F}}$ of fields together with the binary relation symbol $vx \geq vy$. The **theory of valued fields** is the theory of fields (in the language of fields) together with the axioms

(V0) $(\forall y\, vx \geq vy) \Leftrightarrow x = 0$

(VT) $v(x-y) \geq vx \vee v(x-y) \geq vy$

and the axioms which state that the value group is an ordered abelian group:

(VV$\not R$) $\neg(vx < vx)$

(VVT) $vx < vy \wedge vy < vz \Rightarrow vx < vz$

(VVC) $vx < vy \vee vx = vy \vee vx > vy$

(VVG) $vx < vy \Rightarrow vxz < vyz$

(the group axioms for the value group follow from the group axioms for the multiplicative group of the field).

The following facts are well-known; the easy proofs are left to the reader.

Lemma 3.1 *Let (K,v) be a valued field.*

a) For every sentence φ in the language of ordered groups there is a sentence φ' in the language of valued fields such that for every valued field (K,v), φ holds in vK if and only if φ' holds in (K,v).

b) For every sentence φ in the language of rings there is a sentence φ' in the language of valued fields such that for every valued field (K,v), φ holds in Kv if and only if φ' holds in (K,v).

As immediate consequences of this lemma, we obtain:

Corollary 3.2 *If (K,v) and (L,v) are valued fields such that $(K,v) \equiv (L,v)$ in the language of valued fields, then $vK \equiv vL$ in the language of ordered groups, and $Kv \equiv Lv$ in the language of rings (and thus also in the language of fields). The same holds with $\prec$ or $\prec_\exists$ in the place of $\equiv$.*

Corollary 3.3 *If (K,v) is κ-saturated, then so are vK (in the language of ordered groups) and Kv (in the language of fields).*

The property of being henselian is axiomatized by the following axiom scheme:

(HENS) $$vy \geq 0 \wedge \bigwedge_{1\leq i\leq n} vx_i \geq 0 \wedge v(y^n + x_1y^{n-1} + \ldots + x_{n-1}y + x_n) > 0$$
$$\wedge\ v(ny^{n-1} + (n-1)x_1y^{n-2} + \ldots + x_{n-1}) = 0$$
$$\Rightarrow \exists z\ v(y-z) > 0 \wedge z^n + x_1z^{n-1} + \ldots + x_{n-1}z + x_n = 0 \qquad (n \in \mathbb{N})\,.$$

Here we use one of the forms of Hensel's Lemma to characterize henselian fields.

But in view of Theorem 2.14, also the property of being algebraically maximal is easily axiomatized by axiom scheme (HENS) together with the following axiom scheme:

(MAXP) $$\exists y \forall z : v(y^n + x_1y^{n-1} + \ldots + x_{n-1}y + x_n) \geq v(z^n + x_1z^{n-1} + \ldots + x_{n-1}z + x_n) \qquad (n \in \mathbb{N})\,.$$

By the same theorem, the property of being separable-algebraically maximal is axiomatized by axiom scheme (HENS) together with a version of axiom scheme (MAXP) restricted to separable polynomials. This is obtained by adding sentences that state that the coefficient of at least one power y^i for $i > 0$ not divisible by the characteristic of the field is non-zero.

The following was proved by Delon [D] and Ershov [Er]. For the case of valued fields of positive characteristic, we give an alternative proof in [K9].

Lemma 3.4 *The property of being henselian defectless is elementary.*

3.2 The Ax–Kochen–Ershov Principle

Let us now discuss the axiomatization of important properties of valued fields. In particular, let us ask for those properties that a valued field must have if it is an AKE$^\exists$-field. We will need the following well known facts (which were proved, e.g., in L. van den Dries' thesis).

Lemma 3.5 *a) Let $G|H$ be an extension of torsion free abelian groups, and consider it as an extension of $\mathcal{L}_{\rm G}$-structures. If H is existentially closed in G, then G/H is torsion free.*

b) Let $L|K$ be a field extension, and consider it as an extension of $\mathcal{L}_{\rm R}$- or $\mathcal{L}_{\rm F}$-structures. If K is existentially closed in L, then $L|K$ is regular.

An immediate consequence of (3) is the following observation:

Lemma 3.6 *Every AKE$^\exists$-field is algebraically maximal.*

Proof: Let (K, v) be a valued field which admits an immediate algebraic extension (L, v). Then by Lemma 3.5, K is not existentially closed in L. Hence, (K, v) is not existentially closed in (L, v). But $vK = vL$ and $Kv = Lv$, so that the conditions $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$ hold. This shows that (K, v) is not an AKE$^\exists$-field. □

In particular, this lemma shows that every AKE$^\exists$-field must be henselian.

A special case of the AKE$^\exists$ Principle is given if an extension $(L|K,v)$ is immediate. Then, the side conditions are trivially true. We conclude that an AKE$^\exists$-field must in particular be existentially closed in every immediate extension (L,v). (We have used this idea already in the proof of the foregoing lemma.) We can exploit this fact by taking $(\mathfrak{M},v)$ to be a maximal immediate extension of (K,v) and asking which properties of $(\mathfrak{M},v)$ are inherited by (K,v) if $(K,v) \prec_\exists (\mathfrak{M},v)$. We know that $(\mathfrak{M},v)$ has strong properties since it is spherically complete. Nevertheless, we will see later that if (K,v) is henselian of residue characteristic 0, then we even have that $(K,v) \prec (\mathfrak{M},v)$, which means that the elementary properties of $(\mathfrak{M},v)$ are not stronger than that of (K,v). For other classes of valued fields, the situation can be much different. Let us prove that every AKE$^\exists$-field is henselian defectless:

Lemma 3.7 *Let (K,v) be a valued field and assume that there is some maximal immediate extension $(\mathfrak{M},v)$ of (K,v) which satisfies $(K,v) \prec_\exists (\mathfrak{M},v)$. Then (K,v) is henselian defectless. In particular, every AKE$^\exists$-field is henselian defectless.*

Proof: Let $(E|K,v)$ be an arbitrary finite extension. We take $(E|K,v)^*$ to be a κ-saturated elementary extension of $(E|K,v)$. Then (E^*,v^*) and (K^*,v^*) are κ-saturated elementary extensions of (E,v) and (K,v) respectively. We choose κ such that $\kappa \geq |L|^+$. Since by assumption (K,v) is existentially closed in (L,v), Lemma 5.2.1. of [C–K] shows that we can embed (L,v) over (K,v) in (K^*,v^*). We identify it with its image in (K^*,v^*). Since $(E^*|K^*,v^*)$ is an elementary extension of $(E|K,v)$ and $n=[E:K]<\infty$ we have that $[E^*:K^*]=n$ and thus also $[E.L:L]=n$.

The extension $(L|K,v)$ is immediate, and we will prove the same for $(E.L,v^*)|(E,v)$. Since $E.L|L$ is algebraic and $vL=vK$, we know that $v^*(E.L)/vK$ and hence also $v^*(E.L)/vE$ is a torsion group. For the same reason $Lv=Kv$ yields that $(E.L)v|Kv$ and hence also $(E.L)v|Ev$ is algebraic. On the other hand, since (E^*,v^*) is an elementary extension of (E,v) we know by Lemma 3.5 that v^*E^*/vE is torsion free and that Ev is relatively algebraically closed in E^*v. Combining these facts, we get that

$$v^*(E.L)=vE \quad\text{and}\quad (E.L)v=Ev$$

showing that $(E.L,v^*)|(E,v)$ is immediate, as contended.

Since (L,v) is maximal, it is a defectless field (cf. [W], [K2]). Consequently,

$$[E:K]=n=[E.L:L]=(v^*(E.L):vL)\cdot[(E.L)v:Lv]=(vE:vK)\cdot[Ev:Kv]$$

which shows that $(E|K,v)$ is defectless and that the extension of the valuation v from K to E is unique. Since (E,v) was an arbitrary finite extension of (K,v), this shows that (K,v) is a henselian defectless field. □

For the conclusion of this section, let us observe that Lemma 3.1 shows that also the elementary properties of value groups and residue fields of valued fields can be axiomatized in the language of valued fields.

4 Extensions without transcendence defect

Let (K,v) be a henselian defectless field and $(L|K,v)$ an extension without transcendence defect. We choose (K^*,v^*) to be an $|L|^+$-saturated elementary extension of (K,v). Since "henselian" is an elementary property, (K^*,v^*) is henselian like (K,v). Further, it follows from Corollary 3.3 that K^*v^* is an $|Lv|^+$-saturated elementary extension of Kv and that v^*K^* is a $|vL|^+$-saturated elementary extension of vK. Assume that the side conditions $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$ hold. Then by Lemma 5.2.1 of [C–K], there exist embeddings

$$\rho:\ vL \longrightarrow v^*K^*$$

over vK and

$$\sigma:\ Lv \longrightarrow K^*v^*$$

over Kv. Here, the embeddings of value groups and residue fields are understood to be monomorphisms of groups resp. fields. On the other hand, the notation "$\iota:\ (F,v) \longrightarrow (K^*,v^*)$" will indicate an embedding of valued fields, i.e. a valuation preserving monomorphism:

$$\forall x \in F:\ x \in \mathcal{O}_F \Longleftrightarrow \iota x \in \mathcal{O}_{K^*}\ .$$

By definition of "without transcendence defect", every finitely generated subextension $(F|K,v)$ of $(L|K,v)$ is a function field without transcendence defect.

Lemma 4.1 (Embedding Lemma I)
Let (K,v) be a defectless field (the valuation is allowed to be trivial), $(F|K,v)$ a henselian function field without transcendence defect and (K^*,v^*) a henselian extension of (K,v). Assume that vF/vK is torsion free and that $Fv|Kv$ is separable. If $\rho:\ vF \longrightarrow v^*K^*$ is an embedding over vK and $\sigma:\ Fv \longrightarrow K^*v^*$ is an embedding over Kv, then there exists an embedding $\iota:\ (F,v) \longrightarrow (K^*,v^*)$ over (K,v) that respects ρ and σ, i.e. $v^*\iota a = \rho(va)$ and $\iota(a)v^* = \sigma(av)$ for all $a \in F$.

Proof: We choose a transcendence basis $\mathcal{T}$ as in Theorem 1.8. First we will construct the embedding for $K(\mathcal{T})$ and then we will show how to extend it to F.

We choose elements $x'_1,\ldots,x'_r \in K^*$ such that $v^*x'_i = \rho(vx_i)$, $1 \le i \le r$. The values $v^*x'_1,\ldots,v^*x'_r$ are rationally independent over vK since the same holds for their preimages $vx_1,\ldots,vx_r$ and this property is preserved by every monomorphism over vK. Next, we choose elements $y'_1,\ldots,y'_s \in \mathcal{O}_{K^*}^\times$ such that $y'_jv^* = \sigma(y_jv)$, $1 \le j \le s$. The residues $y'_1v^*,\ldots,y'_sv^*$ are algebraically independent over Kv since the same holds for their preimages $y_1v,\ldots,y_sv$ and this property is preserved by every monomorphism over Kv. Consequently, the elements $x'_1,\ldots,x'_r$ and $y'_1,\ldots,y'_s$ as well as the elements $x_1,\ldots,x_r$ and $y_1,\ldots,y_s$ satisfy the conditions of Lemma 2.1. Hence, both sets $\mathcal{T}$ and $\mathcal{T}'$ are algebraically independent over K, so that the assignment

$$x_i \mapsto x'_i\ ,\ y_j \mapsto y'_j\ \ 1 \le i \le r\ ,\ 1 \le j \le s$$

induces an isomorphism $\iota : K(\mathcal{T}) \longrightarrow K(\mathcal{T}')$. Furthermore, for every $f \in K[\mathcal{T}]$, written as in (8),

$$\begin{aligned} v^*\iota f &= \min_k v^*c_k + \sum_{i\in I}\mu_{k,i}v^*x'_i \;=\; \min_k vc_k + \sum_{i\in I}\mu_{k,i}\rho vx_i \\ &= \rho\left(\min_k vc_k + \sum_{i\in I}\mu_{k,i}vx_i\right) \;=\; \rho vf \;, \end{aligned}$$

showing that ι respects the restriction of ρ to $vK(\mathcal{T})$. If $vf = 0$, then

$$fv \;=\; \left(\sum_\ell c_\ell \prod_{j\in J} y_j^{\nu_{\ell,j}}\right) v \;=\; \sum_\ell (c_\ell v)\prod_{j\in J}(y_jv)^{\nu_{\ell,j}}$$

where the sum runs only over those $\ell = k$ for which $\mu_{k,i} = 0$ for all i, and a similar formula holds for $(\iota f)v$. Hence,

$$\begin{aligned} (\iota f)v^* &= \sum_\ell (c_\ell v^*)\prod_{j\in J}(y_jv^*)^{\nu_{\ell,j}} \;=\; \sum_\ell (c_\ell v)\prod_{j\in J}\sigma(y_jv)^{\nu_{\ell,j}} \\ &= \sigma\left(\sum_\ell (c_\ell v)\prod_{j\in J}(y_jv)^{\nu_{\ell,j}}\right) \;=\; \sigma(fv)\;, \end{aligned}$$

showing that ι respects the restriction of σ to $K(\mathcal{T})v$.

By the universal property of henselizations, ι extends to a valuation preserving embedding of $K(\mathcal{T})^h$ in K^* since by hypothesis, K^* is henselian. Since $K(\mathcal{T})^h|K(\mathcal{T})$ is immediate, this embedding trivially also respects the above mentioned restrictions of ρ and σ. Through this embedding, we will from now on identify $K(\mathcal{T})^h$ with its image in K^*. To simplify notation, let us put $L = K(\mathcal{T})^h$.

Now we have to extend ι (which by our identification has become the identity) to an embedding of F in K^* (over L) which respects ρ and σ. This is done as follows. By hypothesis, $F|L$ is finite, tame and unramified. Consequently, $Fv|Lv$ is a finite separable extension, generated by one element, say av. Let $f \in \mathcal{O}_L[X]$ be monic such that its residue polynomial fv is the minimal polynomial of av over Lv; by hypothesis, fv is separable. Hensel's Lemma shows that there exists exactly one root a of f in F having residue av, and exactly one root a' of f in the henselian field K^* having residue $\sigma(av)$. The assignment

$$a \mapsto a'$$

induces an isomorphism $\iota : K(a) \longrightarrow K(a')$ which is valuation preserving since L is henselian. $F|L$ being unramified, $L(a)|L$ is unramified too. Thus ι respects ρ (which after the above identification is the identity). We have to show that ι also respects σ.

Let $n = [L(a) : L]$. Since the elements $1, av, \ldots, (av)^{n-1}$ are linearly independent, the basis $1, a, \ldots, a^{n-1}$ is a valuation basis of $L(a)|L$. Let $g(a) \in L[a]$ where $g \in L[X]$ is of degree $< n$; if $vg(a) = 0$, then $g \in \mathcal{O}_L[X]$ and thus, $g(a)v = (gv)(av)$. In this case,

$$(\iota g(a))v^* = g(a')v^* = (gv)(a'v^*) = (gv)(\sigma(av)) = \sigma((gv)(av)) = \sigma(g(a)v)\;.$$

This proves that ι respects σ.

We have constructed an embedding of $L(a)$ in K^* which respects ρ and σ. But since $F|L$ is a finite tame and unramified extension, we have

$$[F:L]=[Fv:Lv]=[L(a)v:Lv]=[L(a):L]$$

which shows $F=L(a)$, and ι is the required embedding. $\square$

By assumption, $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$, which implies that $vK \prec_\exists vF$ and $Kv \prec_\exists Fv$ because $vF|vK$ is a subextension of $vL|vK$, and $Fv|Kv$ is a subextension of $Lv|Kv$. So we can infer from Lemma 3.5 that the conditions "vF/vK is torsion free" and "$Fv|Kv$ is separable" are satisfied. Hence there is an embedding

$$\iota:\ (F,v)\longrightarrow (K^*,v^*)$$

over K that respects ρ and σ. This shows that (K,v) is existentially closed in (F,v). Since this holds for every finitely generated subextension $(F|K,v)$ of $(L|K,v)$, we have now proved Theorem 1.2.

For further use, we have to make our result more precise:

Lemma 4.2 (Embedding Lemma II)
Let (K,v) be a defectless field (the valuation is allowed to be trivial), $(L|K,v)$ an extension without transcendence defect and (K^*,v^*) an $|L|^+$-saturated henselian extension of (K,v). Assume that vL/vK is torsion free and that $Lv|Kv$ is separable. If

$$\rho:\ vL\longrightarrow v^*K^*$$

is an embedding over vK and

$$\sigma:\ Lv\longrightarrow K^*v^*$$

is an embedding over Kv, then there exists an embedding

$$\iota:\ (L,v)\longrightarrow (K^*,v^*)$$

over K which respects ρ and σ.

Proof: We have already shown that every finitely generated subextension of $(L|K,v)$ can be embedded over (K,v) in (K^*,v^*) respecting both embeddings ρ and σ. Using the saturation property of (K^*,v^*) we have to deduce our assertion from this. To do so, we will work in an extended language $\mathcal{L}'$ consisting of the language $\mathcal{L}_{\rm VF}$ of valued fields together with the predicates

$$\mathcal{P}_\alpha\,,\qquad \alpha\in\rho vL$$
$$\mathcal{Q}_\zeta\,,\qquad \zeta\in\sigma Lv$$

which are interpreted in (K^*, v^*) such that

$$\begin{aligned} \mathcal{P}_\alpha(a) &\iff v^*a = \alpha \\ \mathcal{Q}_\zeta(a) &\iff av^* = \zeta \end{aligned}$$

for all $a \in K^*$ and in (L, v) such that

$$\begin{aligned} \mathcal{P}_\alpha(b) &\iff \rho vb = \alpha \\ \mathcal{Q}_\zeta(b) &\iff \sigma(bv) = \zeta \end{aligned}$$

for all $b \in L$. Note that these interpretations coincide on K.

We show that (K^*, v^*) remains $|L|^+$-saturated in the extended language $\mathcal{L}'$. To this end, we choose a subset $S_v \subset K^*$ of representatives for all values α in ρvL, and a subset $S_r \subset K^*$ of representatives for all residues ζ in σLv. We compute

$$\begin{aligned} |S_v| &= |\rho vL| = |vL| \leq |L| < |L|^+ , \\ |S_r| &= |\sigma Lv| = |Lv| \leq |L| < |L|^+ , \end{aligned}$$

hence $|S_v \cup S_r| < |L|^+$. Consequently, it follows that (K^*, v^*) remains $|L|^+$-saturated in the extended language $\mathcal{L}_{\mathrm{VF}}(S_v \cup S_r)$ (the new constants are interpreted in K^* by the corresponding elements from $S_v \cup S_r$). Now the predicates $\mathcal{P}_\alpha$ and $\mathcal{Q}_\zeta$ become definable in the language $\mathcal{L}_{\mathrm{VF}}(S_v \cup S_r)$. Indeed, if $\alpha \in \rho vL$, then we choose $b_\alpha \in S_v$ such that $vb_\alpha = \alpha$ and define $\mathcal{P}_\alpha(x) :\Leftrightarrow vx = vb_\alpha$. If $\zeta \in \sigma Lv$, then we choose $b_\zeta \in S_r$ such that $b_\zeta v^* = \zeta$ and define $\mathcal{Q}_\zeta(x) :\Leftrightarrow v^*(x - b_\zeta) > 0$. Since (K^*, v^*) is $|L|^+$-saturated in the language $\mathcal{L}_{\mathrm{VF}}(S_v \cup S_r)$, it follows that it is also $|L|^+$-saturated in the language $\mathcal{L}'(S_v \cup S_r)$ and thus also in the language $\mathcal{L}'$, as asserted.

An embedding ι of an arbitrary subextension (L_0, v) of $(L|K, v)$) in (K^*, v^*) over K respects the predicates $\mathcal{P}_\alpha$ and $\mathcal{Q}_\zeta$ if and only if it satisfies, for all $b \in L_0$,

$$\begin{aligned} \rho vb = \alpha \iff (L_0, v) \models \mathcal{P}_\alpha(b) \iff (K^*, v^*) \models \mathcal{P}_\alpha(\iota b) \iff v^*\iota b = \alpha , \\ \sigma(bv) = \zeta \iff (L_0, v) \models \mathcal{Q}_\zeta(b) \iff (K^*, v^*) \models \mathcal{Q}_\zeta(\iota b) \iff \iota bv^* = \zeta , \end{aligned}$$

which expresses the property of ι to respect the embeddings ρ and σ. We know that for every finitely generated subextension of $(L|K, v)$ there exists such an embedding ι. The saturation property of (K^*, v^*) now yields an embedding of (L, v) in (K^*, v^*) over K which respects the predicates and thus the embeddings ρ and σ. This completes the proof of our lemma. □

5 The Relative Embedding Property

Inspired by the assertion of Lemma 4.2, we define a property that will play a key role in our treatment of the model theory of valued fields. Let **C** be a class of valued fields.

We will say that $\mathbf{C}$ has the **Relative Embedding Property**, if the following holds: If $(L, v), (K^*, v^*) \in \mathbf{C}$ and (K, v) is a defectless valued subfield of (L, v) and (K^*, v^*) such that (K^*, v^*) is $|L|^+$-saturated, vL/vK is torsion free and $Lv|Kv$ is separable, and if there are embeddings $\rho : \ vL \longrightarrow v^*K^*$ over vK and $\sigma : \ Lv \longrightarrow K^*v^*$ over Kv, then there exists an embedding $\iota : \ (L, v) \longrightarrow (K^*, v^*)$ over K which respects ρ and σ.

We will show that the Relative Embedding Property implies the following important property. If for every two fields $(L, v), (F, v) \in \mathbf{C}$ and every common defectless subfield (K, v) of (L, v) and (F, v) such that vL/vK is torsion free and $Lv|Kv$ is separable, the side conditions $vL \equiv_{vK} vF$ and $Lv \equiv_{Kv} Fv$ imply that $(L, v) \equiv_{(K,v)} (F, v)$, then we will call $\mathbf{C}$ **relatively subcomplete**. Note that if $\mathbf{C}$ is a relatively subcomplete class of defectless fields, then $\mathbf{C}$ is relatively model complete because by Lemma 3.5, the side conditions $vK \prec vL$ and $Kv \prec Lv$ imply that $(K, v) \prec (L, v)$, and they imply that $vK \equiv_{vK} vL$ and $Kv \equiv_{Kv} Lv$. But relative model completeness is weaker than relative subcompleteness, because $vL \equiv_{vK} vF$ does not imply that $vK \prec vL$, and $Lv \equiv_{Kv} Fv$ does not imply that $Kv \prec Lv$.

The following lemma shows that the Relative Embedding Property is a very powerful property:

Lemma 5.1 *Let $\mathbf{C}$ be an elementary class of defectless valued fields which has the Relative Embedding Property. Then $\mathbf{C}$ is relatively subcomplete and relatively model complete, and the AKE$^\exists$ Principle is satisfied by all extensions $(L|K, v)$ such that both $(K, v), (L, v) \in \mathbf{C}$. If moreover all fields in $\mathbf{C}$ are of fixed equal characteristic, then $\mathbf{C}$ is relatively complete.*

Now we look for a criterion for an elementary class of valued fields to have the Relative Embedding Property. Somehow, we have to improve Embedding Lemma II (Lemma 4.2) to cover the case of extensions $(L|K, v)$ with transcendence defect. Loosely speaking, these contain an immediate part. The idea is to require that this part can be treated separately, that is, that we find an intermediate field $(L', v) \in \mathbf{C}$ such that $(L|L', v)$ is immediate and $(L'|K, v)$ has no transcendence defect. The immediate part has then to be settled by the following auxiliary embedding lemma. Note that by Theorem 2.12, the condition does automatically hold if (K, v) is algebraically maximal.

Lemma 5.2 (Embedding Lemma III)
Let $(K(x)|K, v)$ be a nontrivial immediate extension of valued fields. If x is the limit of a pseudo Cauchy sequence of transcendental type in (K, v), then $(K(x), v)^h$ can be embedded over K in every $|K|^+$-saturated henselian extension $(K, v)^*$ of (K, v).

Note that the lemma fails if the condition "transcendental" on the pseudo Cauchy sequence is omitted, even if we require in addition that (K, v) is henselian. There exist nontrivial finite immediate extensions $(K(x)|K, v)$ of henselian fields. On the other hand, K^* may be a regular extension of K (e.g., this is always the case if $(K, v)^*$ is an elementary extension of (K, v)), and then, $K(x)$ does certainly not admit an embedding over K in K^*.

The model theoretic application of Embedding Lemma III is:

Corollary 5.3 *Let (K,v) be a henselian field and $(K(x)|K,v)$ an immediate extension such that* $\mathrm{appr}\,(x,K)$ *is transcendental. Then $(K,v) \prec_\exists (K(x),v)^h$. In particular, an algebraically maximal field (K,v) is existentially closed in every immediate henselian rational function field $(K(x),v)^h$.*

Proof: Choose $(K,v)^*$ to be a $|K|^+$-saturated elementary extension of (K,v). Since "henselian" is an elementary property, $(K,v)^*$ will also be henselian. Apply Embedding Lemma III and Proposition 5.2.2 of [C–K]. □

Now we are able to give the announced criterion:

Lemma 5.4 *Let* **C** *be an elementary class of valued fields which satisfies*

(CHDL) *every field in* **C** *is henselian defectless,*

(CRAC) *given $(L,v) \in$* **C** *and K' relatively algebraically closed in L such that $Lv|K'v$ is algebraic and vL/vK' is a torsion group, then $(K',v) \in$* **C** *with $Lv = K'v$ and $vL = vK'$,*

(CIMM) *if $(K,v) \in$* **C***, then every immediate henselian function field of transcendence degree 1 over (K,v) is henselian rational.*

Then **C** *has the Relative Embedding Property and is relatively subcomplete and relatively model complete, and the AKE$^\exists$ Principle is satisfied by all extensions $(L|K,v)$ such that both (K,v) and (L,v) are members of* **C**.

6 Model theory of tame fields

In this section we wish to treat the model theory of tame fields. We have already shown in Lemma 2.20 that in positive characteristic, the class of tame fields coincides with the class of algebraically maximal perfect fields. Let us show that the property of being a tame field of fixed residue characteristic is elementary. If the residue characteristic is fixed to 0 then by Lemma 2.16, "tame" is equivalent to "henselian" which is axiomatized by the axiom scheme (HENS). Now assume that the residue characteristic is fixed to be a positive prime. By Lemma 2.20, a valued field of positive residue characteristic is tame if and only if it is an algebraically maximal field having p-divisible value group and perfect residue field. A valued field (K,v) has p-divisible value group if and only if it satisfies the following elementary axiom:

(VGD$_p$) $\forall x\,\exists y :\; vxy^p = 0\;.$

Furthermore, (K,v) has perfect residue field if and only if it satisfies:

(RFD$_p$) $\forall x\,\exists y :\; vx = 0 \;\rightarrow\; v(xy^p - 1) > 0\;.$

Finally, the property of being algebraically maximal is axiomatized by the axiom schemes (HENS) and (MAXP). We summarize: The **theory of tame fields of residue characteristic** 0 is just the theory of henselian fields of residue characteristic 0. If p is a prime, then the **theory of tame fields of residue characteristic** p is the theory of valued fields together with axioms (VGD_p), (RFD_p), (HENS) and (MAXP). Now we also see how we have to proceed to axiomatize the theory of all tame fields. Indeed, for residue characteristic 0 there are no conditions on the value group and the residue field. For residue characteristic $p > 0$, we have to require (VGD_p) and (RFD_p). We can do this by the axiom scheme

(TAD) $\qquad v(p\cdot 1) > 0 \;\rightarrow\; (\mathrm{VGD}_p) \wedge (\mathrm{RFD}_p) \qquad\qquad (p \text{ prime})\,.$

So the **theory of tame fields** is the theory of valued fields together with axioms (TAD), (HENS) and (MAXP).

Recall that a valued field of positive characteristic is tame if and only if it is algebraically maximal and perfect. We have already seen that every $\mathrm{AKE}^\exists$-field must be henselian defectless and in particular algebraically maximal. So this condition is necessary to obtain good model theoretical results. In view of this fact, it is also the model theory of perfect field in positive characteristic that we will develop now.

Let $\mathbf{C}$ be the elementary class of all tame fields. By definition, all tame fields are henselian defectless, so $\mathbf{C}$ satisfies condition (CHDL) of Lemma 5.4. By Lemma 2.23, it also satisfies condition (CRAC). Finally, it satisfies (CIMM) by virtue of Theorem 1.9. Hence, we can infer from Lemma 5.4 and Lemma 5.1:

Theorem 6.1 *The elementary class of tame fields has the Relative Embedding Property and is relatively subcomplete and relatively model complete. Every elementary class of tame fields of fixed equal characteristic is relatively complete.*

Lemma 5.4 does not give the full information about the $\mathrm{AKE}^\exists$ Principle. Therefore, we need the following lemma:

Lemma 6.2 *If Γ is a p-divisible ordered abelian group and $\Gamma \prec_\exists \Delta$, then Γ is also existentially closed in the p-divisible hull of Δ. If k is a perfect field and $k \prec_\exists \ell$, then k is also existentially closed in the perfect hull of ℓ.*

If (K,v) is a tame field and $(L|K,v)$ an extension with $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$, then every maximal purely wild algebraic extension (W,v) of (L,v) is a tame field satisfying $vK \prec_\exists vW$ and $Kv \prec_\exists Wv$.

Proof: By Lemma 5.2.1. of [C–K], $\Gamma \prec_\exists \Delta$ implies that Δ is embeddable over Γ in every $|\Delta|^+$-saturated elementary extension of Γ. Such an elementary extension is p-divisible like Γ. Hence, the embedding can be extended to an embedding of $\frac{1}{p^\infty}\Delta$, which by Lemma 5.2.1. of [C–K] shows that $\Gamma \prec_\exists \frac{1}{p^\infty}\Delta$.

Again by the same lemma, $k \prec_\exists \ell$ implies that ℓ is embeddable over k in every $|\ell|^+$-saturated elementary extension of k. Such an elementary extension is perfect like k.

Hence, the embedding can be extended to an embedding of ℓ^{1/p^∞}, which by Lemma 5.2.1. of [C–K] shows that $k \prec_\exists \ell^{1/p^\infty}$.

By Theorem 2.19, vW is the p-divisible hull $\frac{1}{p^\infty}vL$ of vL, and Wv is the perfect hull Lv^{1/p^∞} of Lv. So our assertion follows since we have just proved that vK (which is itself p-divisible by Lemma 2.20) is existentially closed in $\frac{1}{p^\infty}vL$ and that Kv (which is itself perfect by Lemma 2.20) is existentially closed in the perfect hull Lv^{1/p^∞} of Lv. □

Assume that (K,v) is a tame field and $(L|K,v)$ an extension such that $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$. We choose the tame field (W,v) according to the foregoing lemma. If $(K,v) \prec_\exists (W,v)$, then also $(K,v) \prec_\exists (L,v)$. In view of Lemma 5.4, this proves part a) of Theorem 1.3.

Now let $\mathbf{C}$ be an elementary class of valued fields. We define

$$v\mathbf{C} := \{vK \mid (K,v) \in \mathbf{C}\} \quad \text{and} \quad \mathbf{C}v := \{Kv \mid (K,v) \in \mathbf{C}\}\ .$$

If both $v\mathbf{C}$ and $\mathbf{C}v$ are elementary model complete classes, then the side conditions $vK \prec vL$ and $Kv \prec Lv$ will hold for every two members $(K,v) \subset (L,v)$ of $\mathbf{C}$. Similarly, if $v\mathbf{C}$ and $\mathbf{C}v$ are elementary complete classes, then the side conditions $vK \equiv vL$ and $Kv \equiv Lv$ will hold for all $(K,v),(L,v) \in \mathbf{C}$. So we obtain from the foregoing theorems:

Theorem 6.3 *If $\mathbf{C}$ is an elementary class consisting of tame fields and if $v\mathbf{C}$ and $\mathbf{C}v$ are elementary model complete classes, then $\mathbf{C}$ is model complete. If $\mathbf{C}$ is an elementary class consisting of tame fields of fixed equal characteristic, and if $v\mathbf{C}$ and $\mathbf{C}v$ are elementary complete classes, then $\mathbf{C}$ is complete.*

The converses are true by virtue of Corollary 3.2, provided that $v\mathbf{C}$ and $\mathbf{C}v$ are elementary classes.

Corollary 6.4 *Let $\mathbf{T}$ be an elementary theory consisting of perfect valued fields of equal characteristic. Assume that the value groups and the residue fields of the models of $\mathbf{T}$ form model complete elementary classes. Then the theory $\mathbf{T}^*$ of algebraically maximal valued fields satisfying $\mathbf{T}$ is the model companion of $\mathbf{T}$.*

Proof: It follows from Theorem 6.3 that $\mathbf{T}^*$ is model complete. For every model K of $\mathbf{T}$, any maximal immediate algebraic extension is a model of $\mathbf{T}^*$ (by Lemma 2.20); note that it is an extension of K having the same value group and residue field. □

Note that in the case of positive characteristic, $\mathbf{T}^*$ is in general not a model completion since there exist perfect valued fields of positive characteristic which admit two nonisomorphic maximal immediate algebraic extension, both being models of the model companion. In the case of equal characteristic 0, the algebraically maximal fields are just the henselian fields, and we find that $\mathbf{T}^*$ is a model completion of $\mathbf{T}$.

Elementary classes of tame fields of equal characteristic admit weak prime models if the elementary classes of their value groups and their residue fields do:

Theorem 6.5 *Let $\mathbf{C}$ be an elementary class consisting of tame fields of equal characteristic. Suppose that there exists an infinite cardinal κ, an ordered group $\Gamma \in v\mathbf{C}$ and a field $k \in \mathbf{C}v$, both of cardinality $\leq \kappa$, such that Γ admits an elementary embedding in every κ^+-saturated member of $v\mathbf{C}$ and k admits an elementary embedding in every κ^+-saturated member of $\mathbf{C}v$. Then there exists $(F,v) \in \mathbf{C}$ of cardinality $\leq \kappa$, having value group Γ and residue field k, such that (F,v) admits an elementary embedding in every κ^+-saturated member of $\mathbf{C}$. Moreover, we can assume that (F,v) admits a valuation transcendence basis over its prime field.*

Proof: By Lemma 2.20 we know that k is perfect. Since k is embeddable in every κ^+-saturated member of $\mathbf{C}v$, all of them must have the same characteristic exponent p. Since every κ^+-saturated member of $\mathbf{C}$ has a κ^+-saturated residue field (cf: Corollary 3.3), it follows that every such member is a tame field of characteristic exponent p. Now let $(E,v) \in \mathbf{C}$ such that $vE = \Gamma$, and let $(E,v)^*$ be a κ^+-saturated elementary extension of (E,v). Since $\mathbf{C}$ is an elementary class, we find that $(E,v)^* \in \mathbf{C}$. Consequently, $(E,v)^*$ is a tame field of characteristic exponent p. By Lemma 2.20, its value group is p-divisible. By Corollary 3.2, v^*E^* is an elementary extension of $vE = \Gamma$, which shows that also Γ is p-divisible.

Now by Lemma 2.22, there exists a tame field (F,v) of cardinality at most κ having value group Γ and residue field k and admitting a valuation transcendence basis over its prime field. If (K^*,v^*) is a κ^+-saturated model of $\mathbf{C}$, then v^*K^* and K^*v^* are κ^+-saturated models of $v\mathbf{C}$ and $\mathbf{C}v$ respectively. Hence by hypothesis, there exists an elementary embedding of Γ in v^*K^* over the trivial group $\{0\}$, and an elementary embedding of k in K^*v^* over the prime field k_0 of k. Now k_0 is at the same time the prime field of F and K^*, and the valuation v is trivial on k_0. We have that that vF/vk_0 is torsion free and $Fv|k_0v$ is separable. Now Embedding Lemma II shows the existence of an embedding of (F,v) in (K^*,v^*) over k_0. By virtue of Theorem 6.1, this embedding is elementary (because the embeddings of value group and residue field are). This shows that (F,v) is elementarily embeddable in every κ^+-saturated model of $\mathbf{C}$. This in turn shows that (F,v) is a model of $\mathbf{C}$ and thus a weak prime model of $\mathbf{C}$. □

The weak prime models that we have constructed in the foregoing proof have the special property that they admit a valuation transcendence basis over their prime field. The following corollary confirms the representative role of models with this property.

Corollary 6.6 *For every tame field (L,v) of arbitrary characteristic, there exists a subfield $(K,v) \prec (L,v)$ such that (K,v) admits a valuation transcendence basis over its prime field and $(L|K,v)$ is immediate.*

Proof: According to Lemma 2.25, for every tame field (L,v) there exists a subfield (K,v) of (L,v) admitting a valuation transcendence basis over its prime field, such that $(L|K,v)$ is immediate. In view of Theorem 6.1, the latter fact shows that $(K,v) \prec (L,v)$. □

As a final example, we want to treat here the **theory of tame fields of fixed positive characteristic with divisible value groups and fixed finite residue field**.

Corollary 6.7 *Every elementary class* $\mathbf{C}$ *of tame fields of fixed positive characteristic with divisible value group and fixed residue field* $\mathbb{F}_q$ $(q = p^r)$ *is model complete, complete and decidable. Moreover, it possesses a model having transcendence degree 1 over* $\mathbb{F}_q$ *and admits an elementary embedding in every* $\aleph_1$*-saturated member of* $\mathbf{C}$.

Proof: The theory of divisible ordered abelian groups is model complete, complete and decidable (cf. [Ro], [Ro–Zk]). The same holds trivially for the theory of the finite field $\mathbb{F}_q$ which has only $\mathbb{F}_q$ as a model. Hence, model completeness, completeness and decidability follow readily from Theorem 6.1 and Corollary 1.4. The prime model is constructed as follows: The valued field $(\mathbb{F}_q(t), v_t)$ has value group $\mathbb{Z}$ and residue field $\mathbb{F}_q$. By adjoining suitable roots of t we can build an algebraic extension (F', v_t) with value group $\mathbb{Q}$ and residue field $\mathbb{F}_q$. Now we let (F, v_t) be a maximal immediate algebraic extension of (F', v_t). By Lemma 2.20, it is a tame field. Moreover, it admits $\{t\}$ as a valuation transcendence basis over its prime field. Note that $|F| = \aleph_0$. Since $\mathbb{Q}$ is a prime model of the theory of nontrivial divisible ordered abelian groups, Embedding Lemma II (Lemma 4.2) shows that (F, v_t) admits an embedding in every $\aleph_1$-saturated member of $\mathbf{C}$. By the model completeness that we have already proved, this embedding is elementary. □

7 Separably defectless and separably tame fields

We will first deal with the completion of a valued field.

Theorem 7.1 *Let* (K, v) *be a henselian field. Assume that* $(L|K, v)$ *is a separable subextension of* $(K^c|K, v)$. *Then* (K, v) *is existentially closed in* (L, v). *In particular, every henselian inseparably defectless field is existentially closed in its completion.*

Proof: It suffices to show that (K, v) is existentially closed in every subfield (F, v) of (L, v) which is finitely generated over K. Equivalently, it suffices to show that (K, v) is existentially closed in $(F, v)^h$; note that $(F, v)^h \subset (K, v)^c$ since the completion of a henselian field is again henselian (cf. [W], Theorem 32.19). Take any separating transcendence basis $T = \{x_1, \ldots, x_n\}$ of $F|K$. Then (F, v) lies in the completion of $(K(T), v)$ since it lies in the completion of (K, v). Consequently, also F^h lies in the completion of $K(T)^h$, which is equal to K^c since (K^c, v) is henselian. Now $F^h|K(T)^h$ is a finite separable extension; since a henselian field is separable-algebraically closed in its completion (cf. [W], Theorem 32.19), it must be trivial. That is,

$$(F, v)^h = (K(x_1, \ldots, x_n), v)^h \;,$$

Set $F_0 = K$ and $(F_i, v) = (K(x_1, \ldots, x_i), v)^h$, $1 \leq i \leq n$, where the henselization is taken within F^h. Now it suffices to show that $(F_{i-1}, v) \prec_\exists (F_i, v)$ for $1 \leq i \leq n$.

As x_i is an element of the completion K^c of (F_{i-1}, v) and is transcendental over F_{i-1}, it is the limit of a Cauchy sequence of transcendental type. Hence by Corollary 5.3, $(F_{i-1}, v) \prec_\exists (F_{i-1}(x_i), v)^h$ for $1 \le i \le n$, which in view of $(F_{i-1}(x_i), v)^h = (F_i, v)^h$ proves our assertion.

The second assertion of our theorem follows from the first and the fact that if (K, v) is inseparably defectless, then the immediate extension $K^c|K$ is separable. To show this, take any finite purely inseparable algebraic extension $L|K$; we have to prove that it is linearly disjoint from $K^c|K$. We compute:

$$\begin{aligned} [L:K] &\ge [K^c.L:K^c] \ge (v(K^c.L):vK^c)[(K^c.L)v:K^cv] \\ &= (v(K^c.L):vK)[(K^c.L)v:Kv] \ge (vL:vK)[Lv:Kv] = [L:K]\ , \end{aligned}$$

where the first equality holds because $(K^c|K, v)$ is immediate and the second holds because (K, v) is inseparably defectless. Hence, equality holds everywhere, so that $[K^c.L : K^c] = [L : K]$, as required. □

Since $(K((t)), v_t)$ is henselian, we can choose the henselization $(K(t), v_t)^h$ in $(K((t)), v_t)$. Then $(K((t)), v_t)$ is the completion of both $(K(t), v_t)$ and $(K(t), v_t)^h$. Further, (K, v_t) is trivially valued and thus defectless. By Theorem 1.7, it follows that $(K(t), v_t)^h$ is henselian defectless. Using Theorem 7.1, we conclude:

Theorem 7.2 *Let K be an arbitrary field. Then $(K(t), v_t)^h \prec_\exists (K((t)), v_t)$.*

To give a further application, we need another lemma.

Lemma 7.3 *Let t be transcendental over K and v_t denote the t-adic valuation on $K(t)$. Suppose that K admits a nontrivial henselian valuation v. Then $(K, v) \prec_\exists (K(t), v_t \circ v)^h$.*

Proof: Let (K^*, v^*) be a $|K(t)^h|^+$-saturated elementary extension of (K, v). Then by Corollary 3.3, v^*K^* is a $|vK|^+$-saturated elementary extension of vK. Hence, there exists an element $\alpha \in v^*K^*$ such that $\alpha > vK$. We also have that $(v_t \circ v)t > vK$. Now if $\Gamma \subset \Delta$ is an extension of ordered abelian groups and $\alpha \in \Delta$ such that $\alpha > \Gamma$, then the ordering on $\mathbb{Z}\alpha + \Gamma$ is uniquely determined. Indeed, $\mathbb{Z}\alpha + \Gamma$ is the lexicographic product $\mathbb{Z}\alpha \amalg \Gamma$. So we see that the assignment $(v_t \circ v)t \mapsto \alpha$ induces an embedding of $(v_t \circ v)K(t) = \mathbb{Z}(v_t \circ v)t \amalg vK$ in v^*K^* over vK as ordered groups. Now choose $t^* \in K^*$ such that $v^*t^* = \alpha$. As $(v_t \circ v)t$ and α are not torsion elements over vK, Lemma 2.1 shows that the assignment $t \mapsto t^*$ induces an embedding of $(K(t), v_t \circ v)$ in (K^*, v^*) over K. Since (K, v) is henselian, so is the elementary extension (K^*, v^*). By the universal property of the henselization, the embedding can thus be extended to an embedding of $(K(t), v_t \circ v)^h$ in (K^*, v^*). By Lemma 5.2.2. of [C–K], this gives our assertion. □

Now we are able to prove:

Theorem 7.4 *If the field K admits a nontrivial henselian valuation, then $K \prec_\exists K((t))$ (as fields).*

Proof: Let v be the nontrivial valuation on K such that (K,v) is henselian. By Lemma 7.3, we have that $(K,v) \prec_\exists (K(t), v_t \circ v)^h$. By the foregoing lemma, $K((t))|K(t)^h$ is separable. Since $(K((t)), v_t)$ is the completion of $(K(t), v_t)$, it follows that $(K((t)), v_t \circ v)$ is the completion of $(K(t), v_t \circ v)$. Hence, Theorem 7.1 shows that $(K(t), v_t \circ v)^h \prec_\exists (K((t)), v_t \circ v)$. It follows that $(K,v) \prec_\exists (K((t)), v_t \circ v)$. In particular, $K \prec_\exists K((t))$, as asserted. □

We turn to extensions without transcendence defect and prove an analogue of Theorem 1.2. We do not know whether the cofinality condition can be dropped.

Theorem 7.5 *Let (K,v) be a henselian separably defectless field and $(L|K,v)$ a separable extension without transcendence defect. Let vK be cofinal in vL. Then $(L|K,v)$ satisfies the AKE$^\exists$ Principle.*

Proof: Assume that $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$. Since vK is supposed to be cofinal in vL, we know that $(K,v)^c$ is contained in $(L,v)^c$. The compositum $(L.K^c, v)$, taken in the completion $(L,v)^c$, is an immediate extension of (L,v). Thus, $vK^c = vK \prec_\exists vL = vL.K^c$ and $K^c v = Kv \prec_\exists Lv = (L.K^c)v$. Since (K,v) is a henselian separably defectless field, $(K,v)^c$ is a henselian defectless field. As $(L|K,v)$ is an extension without transcendence defect, the same holds for $(L.K^c|K^c, v)$. By Theorem 1.2, it follows from our side conditions that

$$(K^c, v) \prec_\exists (L.K^c, v) .$$

Let us now take an $|L.K^c|^+$-saturated elementary extension $(K^c|K,v)^*$ of the valued field extension $(K^c|K,v)$. We note that $(K^c,v)^*$ is a subfield of the completion of $(K,v)^*$ since the property of K to be dense in K^c is elementary in the language of valued fields with the predicate $\mathcal{P}$ for the subfield:

$$\forall x \forall y \exists z : \ \mathcal{P}(z) \wedge (\, y \neq 0 \ \rightarrow \ v(x-z) > vy \,)$$

expresses this property.

Since $(K^c, v) \prec_\exists (L.K^c, v)$, Lemma 5.2.1. of [C–K] shows that $(L.K^c, v)$ can be embedded over K^c in $(K^c, v)^*$ and can thus be considered to be a subfield of the completion of $(K,v)^*$, and so the same holds for the smaller field $(L.K^*, v)$. Since $L|K$ is assumed to be separable, it follows that also $L.K^*|K^*$ is separable. Now Theorem 7.1 shows that

$$(K,v)^* \prec_\exists (L.K^*, v) .$$

Since $(K,v) \prec (K,v)^*$, we obtain that $(K,v) \prec_\exists (L.K^*, v)$, which yields that $(K,v) \prec_\exists (L,v)$, as asserted. □

We can now prove part b) of Theorem 1.3:

Theorem 7.6 *Let (K,v) be a separably tame field. Then every separable extension satisfies the AKE$^\exists$ Principle.*

Proof: Assume that $(L|K,v)$ is a separable extension with $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$. The perfect hull K^{1/p^∞} of K admits a unique extension v of the valuation of K, and with this valuation it is a subextension of the completion of K, according to Lemma 2.28. By Lemma 2.29, $(K^{1/p^\infty},v)$ is a tame field. Both K^{1/p^∞} and $L.K^{1/p^\infty}$ are valued subfields of the perfect hull $(L^{1/p^\infty},v)$ of (L,v), whose value group is the p-divisible hull of vL and whose residue field is the perfect hull of Lv. Now $vK^{1/p^\infty} = vK$ is p-divisible and $Kv^{1/p^\infty} = Kv$ is perfect, hence by our side conditions, we obtain that $vK^{1/p^\infty} \prec_\exists v(L.K^{1/p^\infty})$ and $Kv^{1/p^\infty} \prec_\exists (L.K^{1/p^\infty})v$ (cf. Lemma 6.2). According to the AKE$^\exists$ Principle for tame fields (Theorem 1.3), this yields that

$$(K^{1/p^\infty},v) \prec_\exists (L.K^{1/p^\infty},v)\ .$$

Now take a $|L.K^{1/p^\infty}|^+$-saturated elementary extension $(K^{1/p^\infty}|K,v)^*$ of $(K^{1/p^\infty}|K,v)$. From this point on, the proof is just an analogue of the proof of the foregoing theorem.

□

Related to this result are results of F. Delon [D]. She showed that the **elementary class of algebraically maximal Kaplansky fields of fixed p-degree** is relatively complete. Adding predicates to the language of valued fields which guarantee that every extension is separable, she also obtained relative model completeness. Similar results can be derived for separably tame fields of fixed p-degree.

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