A COMMON GENERALIZATION OF METRIC, ULTRAMETRIC AND TOPOLOGICAL FIXED POINT THEOREMS

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Abstract. We present a general fixed point theorem which can be seen as the quintessence of the principles of proof for Banach's Fixed Point Theorem, ultrametric and certain topological fixed point theorems. It works in a minimal setting, not involving any metrics. We demonstrate its applications to the metric, ultrametric and topological cases, and to ordered abelian groups and fields.

1. Introduction

What is the common denominator of Banach's Fixed Point Theorem and its ultrametric and topological analogues as developed in [6, 7, 8, 5] and in [10]? Is there a general principle of proof that works for all of these worlds, the (ordinary) metric, ultrametric and topological, and beyond? In this paper, we give an answer to these questions. We draw our inspiration from the notions of "ball" and "spherical completeness" that are used in the ultrametric world.

S. Priess' paper [6] in which she first proved a fixed point theorem for ultrametric spaces initiated an interesting development that led to a better understanding of important theorems in valuation theory and to new results (see, e.g., [KU4]). This was achieved by extracting the underlying principle of the proof of Hensel's Lemma through abstracting from the algebraic operations and only considering the ultrametric induced by the valuation. In this paper we push this development one step further by extracting the underlying principle of various fixed point theorems. In this way, a general framework is set up that helps understand these theorems in a more conceptual manner and to transfer ideas from one world to the other by analogies (as we will demonstrate, for instance, for the topological fixed point theorem we consider).

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The general framework also helps to make the use of fixed point theorems available to situations that are difficult or even impossible to subsume under the above mentioned settings. While investigating spaces of real places, we found that in certain algebraic entities, it may be much easier and more natural to define "balls" than to define the "distance" between two elements. For example, if we are dealing with quotient topologies, like in the case of spaces of real places where the topology is induced by the Harrison topology of spaces of orderings, balls come up naturally as images of certain open or closed sets in the inducing topology. Therefore, we will work with **ball spaces** $(X,\mathcal{B})$, that is, sets X with a set $\mathcal{B}$ of distinguished nonempty subsets, which we call **balls**. We require no further structure on these spaces. We do not even need a topology generated in some way by the balls. But let us mention that the way we formulate our theorems, the balls should be considered closed, rather than open, in such a topology, because singletons appear and are important. One can reformulate everything in an "open ball" approach, but this makes the exposition less elegant.

We found the idea of centering the attention on balls, rather than metrics, in the paper [2]. But there, ball spaces carry much more structure and the conditions for a fixed point theorem are unnecessarily restrictive.

We will now state our most general fixed point theorem for ball spaces. We need two notions. A **nest of balls** in $(X,\mathcal{B})$ is a nonempty collection of balls in $\mathcal{B}$ that is totally ordered by inclusion. If $f : X \to X$ is a function, then a ball B will be called f-contracting it it is either a singleton containing a fixed point or satisfies $f(B) \subsetneq B$.

Theorem 1. *Take a function f on a ball space $(X,\mathcal{B})$ which satisfies the following conditions:*
(C1) *there is at least one f-contracting ball,*
(C2) *for every f-contracting ball $B \in \mathcal{B}$, the image $f(B)$ contains an f-contracting ball,*
(C3) *the intersection of every nest of f-contracting balls contains an f-contracting ball.*
Then f admits a fixed point.

We can obtain uniqueness of the fixed point by strengthening the hypothesis:

Theorem 2. *Take a function f on a ball space $(X,\mathcal{B})$ which satisfies the following conditions:*
(CU1) *X is an f-contracting ball,*
(CU2) *for every f-contracting ball $B \in \mathcal{B}$, the image $f(B)$ is again an f-contracting ball,*
(CU3) *the intersection of every nest of f-contracting balls is again an f-contracting ball.*
Then f has a unique fixed point.

These theorems will be proved in Section 2.

In [10] the authors show that every "J-contraction" on a connected compact Hausdorff space X has a unique fixed point. Using the inspiration from our general framework, we obtain the following strong generalization:

Theorem 3. *Take a compact space X and a closed function $f : X \to X$. If every closed set in X contains a closed f-contracting subset, then f has a fixed point in X. If every closed set in X is f-contracting, then f has a unique fixed point in X.*

This theorem and the theorem of [10] are corollaries to Theorems 1 and 2. We will show this in Section 4, where we will also present another version that is directly related to Theorem 4 below.

For most applications of these theorems, it is an advantage to have a handy criterion for the existence of the f-contracting balls. From classical fixed point theorems we know the assumption that the function f be strictly contracting. But we have learnt from the ultrametric case that if one does not insist on uniqueness, one can relax the conditions: a function does not need to be strictly contracting on the whole space, but only on the orbits of its elements (and simply contracting otherwise). While this relaxation makes the formulation of the conditions a bit longer, it should be noted that it is important for many applications in which the function under consideration fails for natural reasons to be strictly contracting on the whole space. The following way to present a fixed point theorem may seem unusual, but it turns out to be very close to several applications as it encodes (a weaker form of) the property "strictly contracting on orbits".

Consider a function $f : X \to X$. We will write fx for $f(x)$ and $f^i x$ for the image of x under the i-th iteration of f, that is, $f^0 x = x$, $f^1 x = f(x)$ and $f^{i+1} x = f(f^i x)$. The function f will be called **strongly contracting on orbits** if there is a function

$$X \ni x \mapsto B_x \in \mathcal{B}$$

such that for all $x \in X$, the following conditions hold:

(SC1) $x \in B_x$,

(SC2) $B_{fx} \subseteq B_x$, and if $x \neq fx$, then $B_{f^i x} \subsetneq B_x$ for some $i \geq 1$,

Note that (SC1) and (SC2) imply that $f^i x \in B_x$ for all $i \geq 0$.

We will say that a nest of balls $\mathcal{N}$ is an **f-nest** if $\mathcal{N} = \{B_x \mid x \in S\}$ for some set $S \subseteq X$ that is closed under f. Now we can state our third main theorem:

Theorem 4. *Take a function f on a ball space which is strongly contracting on orbits. If for every f-nest $\mathcal{N}$ in this ball space there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$, then f has a fixed point.*

Theorem 4 does not deal with the question of uniqueness of fixed points; this question is answered in the particular applications by additional arguments that are often very easy.

The condition about the intersection of an f-nest is not needed for Banach's Fixed Point Theorem and may therefore appear alien to readers who are not familiar with the ultrametric case. But there, as in the case of non-archimedean ordered groups and fields, one has to deal with jumps that one could intuitively think of as being a "non-archimedean" or "non-standard" phenomenon. The obstruction is that the intersection of an infinite nest of balls we have constructed may contain more than one element, at which point we have to iterate the construction. The mentioned condition makes this work.

The condition on the intersection of f-nests implies that in particular, they are not empty. This reminds of a similar property of ultrametric spaces, and we take over the corresponding notion. The ball space $(X,\mathcal{B})$ will be called **spherically complete** if every nonempty nest of nonempty balls has a nonempty intersection.

To illustrate the flexibility of the concepts we have introduced and the above explained idea of making fixed point theorems available to totally new settings, we state the following easy but useful result:

Proposition 5. *Take two ball spaces $(X_1,\mathcal{B}_1)$ and $(X_2,\mathcal{B}_2)$ and a function $f : X_1 \to X_2$. Suppose that the preimage of every ball in $\mathcal{B}_2$ is a ball in $\mathcal{B}_1$. If $\mathcal{N}$ is a nest of balls in $(X_2,\mathcal{B}_2)$, then the preimages of the balls in $\mathcal{N}$ form a nest of balls in $(X_1,\mathcal{B}_1)$. If $(X_1,\mathcal{B}_1)$ is spherically complete, then so is $(X_2,\mathcal{B}_2)$.*

In several applications, and in particular in the ultrametric setting, the function under consideration has in a natural way stronger properties than we have used so far. What we have asked for one element in the intersection of an f-nest is often satisfied by every element in the intersection. Therefore, it seems convenient to introduce a notion which reflects this property and in this way to separate it from the condition that the intersections is nonempty. The function f will be called **self-contractive** if in addition to (SC1) and (SC2), it satisfies:

(SC3) if $\mathcal{N}$ is an f-nest and if $z \in \bigcap \mathcal{N}$, then $B_z \subseteq \bigcap \mathcal{N}$.

Self-contractive functions will appear in the hypothesis of Theorem 7, in Theorem 11, in Theorem 12, and in the proof of Banch's Fixed Point Theorem. The following fixed point theorem is an easy corollary to Theorem 4:

Theorem 6. *Every self-contractive function on a spherically complete ball space has a fixed point.*

For the proof of Theorems 1, 2 and 4, see Section 2. In Section 3, we state two general attractor theorems. Section 4 is devoted to topological fixed point theorems. In Section 5, we show how to derive ultrametric fixed point theorems, and in Section 6, we discuss ultrametric attractor theorems. In Section 7 we then discuss valued fields that are complete by stages, a notion introduced by P. Ribenboim in [9]. We use Theorem 4 for a quick proof of a fixed point theorem that works in such fields (Theorem 16). This theorem can be used to show that such fields are henselian. Its application to the proof of Hensel's Lemma provides an example for a case where one does not have in any natural way a function that is strictly contracting on all of the space. Note also that the particularly weak form that we have chosen for (SC2) comes in very handy for the formulation of Theorem 16.

In Section 8 we discuss how to derive Banach's Fixed Point Theorem. Our aim is not to provide a new proof of this theorem; in contrast to our other applications, the existing proofs in this case are much shorter. Our aim here is to show how to convert the problem from metric to ball space and to pave the way for other settings where order balls are used. In particular, we will discuss their use for the case of nonarchimedean ordered groups and fields in Section 9. That case provides examples for the flexibility of our notion of ball space, as we will work with proper subsets of the collection of all order balls, or with hybrid ball spaces that contain order balls as well as ultrametric balls.

2. Proof of the fixed point theorems for ball spaces

Proof of Theorem 1:
The set of all nests consisting of f-contracting balls is partially ordered by inclusion. There is at least one such nest since by condition (C1), there is at least one f-contracting ball. Further, the union over an ascending chain of nests consisting of f-contracting balls is again such a nest (observe that the cardinality of this union is bounded by the cardinality of the power set of X, so the union is a set). Hence by Zorn's Lemma, there is a maximal nest $\mathcal{N}$ consisting of f-contracting balls. By condition (C3), $\bigcap \mathcal{N}$ contains an f-contracting ball B. Suppose this ball is not a singleton. But then by condition (C2), $f(B)$ contains an f-contracting ball B'. Since $B' \subseteq f(B) \subsetneq B$, $\mathcal{N} \cup \{B'\}$ is then a nest that properly contains $\mathcal{N}$, which contradicts the maximality. We find that B must be a singleton consisting of a fixed point. □

Proof of Theorem 2:
Using conditions (CU1), (CU2), (CU3) and transfinite induction, we build a nest $\mathcal{N}$ consisting of f-contracting balls as follows. We set $\mathcal{N}_0 := \{X\}$. Having constructed $\mathcal{N}_\nu$ for some ordinal ν with smallest f-contracting ball $B_\nu \in \mathcal{N}_\nu$, we set $B_{\nu+1} := f(B_\nu)$ and $\mathcal{N}_{\nu+1} := \mathcal{N}_\nu \cup \{B_{\nu+1}\}$. If λ is a limit ordinal and we have constructed $\mathcal{N}_\nu$ for all $\nu < \lambda$, we observe that the union over all $\mathcal{N}_\nu$ is a nest $\mathcal{N}'_\lambda$. We set $B_\lambda := \bigcap \mathcal{N}'_\lambda$ and $\mathcal{N}_\lambda := \mathcal{N}'_\lambda \cup \{B_\lambda\}$.

If B_ν is not a singleton, then $B_{\nu+1} \subsetneq B_\nu$. Hence there must be an ordinal ν of cardinality at most that of X such that $B_{\nu+1} = B_\nu$. But this only happens if B_ν is a singleton consisting of a fixed point x. If $x \neq y \in X$, then $y \notin B_\nu$ which means that there is some $\mu < \nu$ such that $y \in B_\mu$, but $y \notin B_{\mu+1} = f(B_\mu)$. This shows that y cannot be a fixed point of f. Therefore, x is the unique fixed point of f. □

Theorem 4 can be derived from Theorem 1. But as it takes essentially the same effort, we will give a proof along the lines of the proof of Theorem 1.

Proof of Theorem 4:
Take a function f on the ball space $(X,\mathcal{B})$ which is contractive on orbits. For every $x \in X$, the set $\{B_{f^ix} \mid i \geq 0\}$ is an f-nest. The set of all f-nests is partially ordered by inclusion. The union over an ascending chain of f-nests is again an f-nest. Hence by Zorn's Lemma, there is a maximal f-nest $\mathcal{N}$. By the assumption of Theorem 4, there is some $z \in \bigcap\mathcal{N}$ such that $B_z \subseteq \bigcap\mathcal{N}$. We wish to show that z is a fixed point of f. If we would have that $z \neq fz$, then by (SC2), $B_{f^iz} \subsetneq B_z \subseteq \bigcap\mathcal{N}$ for some $i \geq 1$, and the f-nest $\mathcal{N} \cup \{B_{f^kz} \mid k \in \mathbb{N}\}$ would properly contain $\mathcal{N}$. But this would contradict the maximality of $\mathcal{N}$. Hence, z is a fixed point of f. □

3. General attractor theorems

Let us derive from Theorem 6 an attractor theorem which is modeled after the ultrametric attractor theorem in [4]. We consider two ball spaces $(X,\mathcal{B})$ and $(X',\mathcal{B}')$ and a function $\varphi : X \to X'$. Take an element $z' \in X'$. If there is a function $f : X \to X$ which is strongly contracting on orbits, and a function

$$X \ni x \mapsto B'_x \in \mathcal{B}'$$

such that for all $x \in X$, the following conditions hold:

(AT1) $z' \in B'_x$ and $\varphi(B_x) \subseteq B'_x$,
(AT2) if $\varphi(x) \neq z'$, then $B'_{f^ix} \subsetneq B'_x$ for some $i \in \mathbb{N}$,

then z' will be called a **weak f-attractor for** φ. If in addition f is self-contractive, then z' will be called an **attractor for** φ.

Theorem 7. (Attractor Theorem 1)
Take a function $\varphi : X \to X'$ and an attractor $z' \in X'$ for φ. If $(X,\mathcal{B})$ is spherically complete, then $z' \in \varphi(X)$.

Indeed, by Theorem 6, f has a fixed point z. But by condition (AT2), $fz = z$ implies that $\varphi(z) = z'$. The following version of the Attractor Theorem follows in a similar way from Theorem 4:

Theorem 8. (Attractor Theorem 2)
Take a function $\varphi : X \to X'$ and a weak f-attractor $z' \in X'$ for φ. If for every f-nest $\mathcal{N}$ in $(X,\mathcal{B})$ there is some $z \in \bigcap\mathcal{N}$ such that $B_z \subseteq \bigcap\mathcal{N}$, then $z' \in \varphi(X)$.

4. Fixed point theorems for topological spaces

In this section, we consider compact topological spaces X with functions $f : X \to X$. We note:

Lemma 9. *Every compact space X together with any family of nonempty closed subsets is a spherically complete ball space.*

Proof: A nest $\mathcal{N}$ of nonempty closed subsets of X is a centered system in the sense of [1, Proposition 2, p. 57]. By this proposition, the intersection of $\mathcal{N}$ is nonempty. □

In view of this lemma, we will take $\mathcal{B}$ to be the set of all nonempty closed subsets of X. For the first theorem we wish to discuss, we assume that X is compact, Hausdorff and connected. An open cover $\mathcal{U}$ of X is said to be *J-contractive for f* if for every $U \in \mathcal{U}$ there is $U' \in \mathcal{U}$ such that $f(\operatorname{cl} U) \subseteq U'$, where $\operatorname{cl} U$ denotes the closure of U. The function $f : X \to X$ is called *J-contraction* if every open cover $\mathcal{U}$ has a finite J-contractive open refinement $\mathcal{V}$ for f. For every non-connected compact Hausdorff space X there is a J-contraction of X which has no fixed points (cf. [10, Proposition 3, p. 553]); therefore, the approach using J-contractions only works for connected compact Hausdorff spaces. We cite two important facts about a J-contraction f on a connected compact Hausdorff space X:

(J1) If B is a closed subset of X with $f(B) \subseteq B$, then the restriction of f to B is also a J-contraction ([10, Proposition 1, p. 552]);

(J2) If f is onto, then $|X| = 1$ ([10, Proposition 4, p. 554]).

The following theorem is proved in [SWJ]:

Theorem 10. *Take a connected compact Hausdorff space X and a continuous J-contraction $f : X \to X$. Then f has a unique fixed point.*

We wish to deduce this theorem from Theorem 2. We observe that by (J1) and (J2), every closed subset B of X such that $f(B) \subseteq B$ is an f-contracting ball; indeed, since by (J1) f is a J-contraction on B, (J2) shows that either f is not onto, or B is a singleton $\{x\}$ and since $fx \in f(B) \subseteq B$, we have that $fx = x$. In particular, X is an f-contracting ball. If B is an f-contracting ball, hence a closed subset of X, then also $f(B)$ is closed since f is a closed function, being continuous on a compact Hausdorff space. Since $f(B) \subseteq B$, we also have that $f(f(B)) \subseteq f(B)$, and therefore, $f(B)$ is an f-contracting ball. Finally, the intersection $\bigcap \mathcal{N}$ of any nest $\mathcal{N}$ of f-contracting balls is closed and satisfies $f(\bigcap \mathcal{N}) \subseteq \bigcap \mathcal{N}$; therefore, it is also an f-contracting ball. Now Theorem 10 follows from Theorem 2. □

Theorem 3 is easily derived from Theorems 1 and 2. Take a compact space X and a closed function $f : X \to X$. Assume first that every closed set in X contains a closed f-contracting subset. Since X is closed, this implies that condition (C1) is satisfied. If B is an f-contracting closed set in X, then $f(B)$ is closed since f is a closed function. Hence by assumption, it contains an f-contracting closed set, so condition (C2) is satisfied. The intersection of a nest of f-contracting closed sets is also closed; hence by assumption, it contains an f-contracting closed set. So condition (C3) is satisfied, and Theorem 1 shows that f has a fixed point.

Now assume that every closed set in X is f-contracting. Then X is f-contracting, for every f-contracting closed set B also $f(B)$ is closed and f-contracting, and the closed intersection of a nest of f-contracting closed sets is also f-contracting. Therefore, Theorem 2 shows that f has a unique fixed point. □

The next theorem shows how to apply Theorem 4 to toplogical spaces.

Theorem 11. *Take a compact space X and a closed function $f : X \to X$. Assume that for every $x \in X$ with $fx \neq x$ there is a closed subset B of X such that $x \in B$ and $x \notin f(B) \subseteq B$. Then f has a fixed point in B. Moreover, f is self-contractive, and for every $x \in X$ with $fx \neq x$ there is a smallest closed subset B of X such that $x \in B$ and $x \notin f(B) \subseteq B$.*

Proof: For every $x \in X$ we consider the following family of balls:

$$\mathfrak{B}_x := \{B \mid B \text{ closed subset of } X,\, x \in B \text{ and } f(B) \subseteq B\}.$$

Note that $\mathfrak{B}_x$ is nonempty because it contains X. We define

$$B_x := \bigcap \mathfrak{B}_x \,.$$

We see that $x \in B_x$ and that $f(B_x) \subseteq B_x$. For every $B \in \mathfrak{B}_x$ we have that $fx \in B$ and therefore, $B \in \mathfrak{B}_{fx}$. Hence we find that $B_{fx} \subseteq B_x$.

Assume that $fx \neq x$. Then by hypothesis, there is a closed set B in X such that $x \in B$ and $x \notin f(B) \subseteq B$. Since f is a closed function, $f(B)$ is closed. Moreover, $f(f(B)) \subseteq f(B)$ and $fx \in f(B)$, so $f(B) \in \mathfrak{B}_{fx}$. Since $x \notin f(B)$, we conclude that $B_{fx} \subsetneqq B_x$. We have now proved that f is strongly contracting on orbits.

Observe that $B \in \mathfrak{B}_x$, which shows that $x \notin f(B_x)$. Further, B_x is closed, being the intersection of closed sets. Since $f(B_x) \subseteq B_x$, we find that $B_x \in \mathfrak{B}_x$. This in turn shows that B_x is the smallest closed set in X such that $x \in B$ and $x \notin f(B) \subseteq B$.

Take an f-nest $\mathcal{N}$. Lemma 9 shows that $\bigcap \mathcal{N}$ is nonempty. Take any $z \in \bigcap \mathcal{N}$. Choose an arbitrary $B_x \in \mathcal{N}$. Then $z \in B_x$. By assumption, there is a closed set B in X such that $z \in B$ and $z \notin f(B) \subseteq B$. Set $B' := B_x \cap B$; then $z \notin f(B') \subseteq B'$. Being the intersection of two closed sets, B' is also closed. Thus, $B' \in \mathfrak{B}_z$. We see that $B_z \subseteq B' \subseteq B_x$. Therefore, $B_z \subseteq \bigcap \mathcal{N}$. We have proved that f is self-contractive.

Theorem 11 now follows from Theorem 4. □

5. Ultrametric fixed point theorems

Let (X, d) be an ultrametric space. That is, d is a function from $X \times X$ to a partially ordered set Γ with smallest element 0, satisfying that for all $x, y, z \in X$ and all $\gamma \in \Gamma$,

(U1) $d(x, y) = 0$ if and only if $x = y$,
(U2) if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$,
(U3) $d(x, y) = d(y, x)$ (symmetry).

(U2) is the ultrametric triangle law; if Γ is totally ordered, it can be replaced by

(UT) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

We obtain the **ultrametric ball space** $(X, \mathcal{B}_u)$ from (X, d) by taking $\mathcal{B}_u$ to be the set of all

$$B(x, y) := \{z \in X \mid d(x, z) \leq d(x, y)\} .$$

It follows from the ultrametric triangle law that $B(x, y) = B(y, x)$ and that

(1) $\quad B(t, z) \subseteq B(x, y)$ if and only if $t \in B(x, y)$ and $d(t, z) \leq d(x, y)$.

In particular,

$$B(t, z) \subseteq B(x, y) \quad \text{if } t, z \in B(x, y) .$$

Two elements γ and δ of Γ are **comparable** if $\gamma \leq \delta$ or $\gamma \geq \delta$. Hence if $d(x, y)$ and $d(y, z)$ are comparable, then $B(x, y) \subseteq B(y, z)$ or $B(y, z) \subseteq B(x, y)$. If $d(y, z) < d(x, y)$, then in addition, $x \notin B(y, z)$ and thus, $B(y, z) \subsetneq B(x, y)$. We note:

(2) $$d(y, z) < d(x, y) \Longrightarrow B(y, z) \subsetneq B(x, y) .$$

If Γ is totally ordered and B_1 and B_2 are any two balls with nonempty intersection, then $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

The ultrametric space (X, d) is called **spherically complete** if the corresponding ball space is spherically complete. The following theorem ($i = 1$ in (3)) appeared in [8]:

Theorem 12. (Strong Ultrametric Fixed Point Theorem) *Take a spherically complete ultrametric space (X, d) and a function $f : X \to X$. Assume that f satisfies, for all $x, z \in X$:*

(3) $$x \neq fx \Longrightarrow \exists i \geq 1 : d(f^i x, f^{i+1} x) < d(x, fx)$$

(4) $$d(z, fx) \leq d(fx, f^2 x) \Longrightarrow d(z, fz) \leq d(x, fx) .$$

Then f has a fixed point.

Proof: Our theorem follows from Theorem 6 once we have shown that f is self-contractive on the ball space $(X, \mathcal{B}_u)$. We define

(5) $$B_x := B(x, fx)$$

and observe that $x \in B_x$. Taking $z = fx$ in (4), we find that $d(fx, f^2x) \leq d(x, fx)$. Hence by (1), $B_{fx} = B(fx, f^2x) \subseteq B(x, fx) = B_x$. By induction on i it follows that $f^ix \in B_x$. By (3), $d(f^ix, f^{i+1}x) < d(x, fx)$ for some $i \geq 1$. Then by (2), we have that $B_{f^ix} = B(f^ix, f^{i+1}x) \subsetneq B(x, fx) = B_x$. So we have proved that f satisfies (SC1) and (SC2).

To show that also (SC3) holds, we take an f-nest $\mathcal{N}$ and any $z \in \bigcap \mathcal{N}$. We have to show that $B_z \subseteq \bigcap \mathcal{N}$, that is, $B_z \subseteq B_x$ for all $B_x \in \mathcal{N}$. Since $z \in \bigcap \mathcal{N} \subseteq B_{fx} = B(fx, f^2x)$, we have that $d(z, fx) \leq d(fx, f^2x)$. By (4), this implies that $d(z, fz) \leq d(x, fx)$. Since we know that $z \in B_x$, (1) now shows that $B_z = B(z, fz) \subseteq B(x, fx) = B_x$. □

A function $f : X \to X$ is called **contracting** if $d(fx, fy) \leq d(x, y)$ for all $x, y \in X$. It is shown in [8] that the following theorem (in the case of $i = 1$ in (3)) follows from Theorem 12:

Theorem 13. (Ultrametric Fixed Point Theorem)
Every contracting function on a spherically complete ultrametric space which satisfies (3) has a fixed point.

This theorem follows directly from Theorem 6 by way of the following result:

Lemma 14. *Take a contracting function f on an ultrametric space (X, d) and define the balls B_x as in (5). Then f satisfies (SC3), and $f(B_x) \subseteq B_x$ and $B_{fx} \subseteq B_x$ for all $x \in X$. If f also satisfies (3), then it is self-contractive.*

Proof: Take $z \in B_x$, that is, $d(x, z) \leq d(x, fx)$. Since f is contracting, we have that $d(fx, fz) \leq d(x, z) \leq d(x, fx)$. Together with the trivial inequality $d(x, fx) \leq d(x, fx)$ and the ultrametric triangle law, this yields that $d(x, fz) \leq d(x, fx)$. Together with $d(z, x) = d(x, z) \leq d(x, fx)$ and the ultrametric triangle law, this yields that $d(z, fz) \leq d(x, fx)$. Now (1) shows that $B_z = B(z, fz) \subseteq B(x, fx) = B_x$. We obtain that $fz \in B_x$, and as $z \in B_x$ was arbitrary, this shows that $f(B_x) \subseteq B_x$. Taking $z = fx \in B_x$, we find that $B_{fx} \subseteq B_x$.

If $\mathcal{N}$ is an f-nest and $z \in \bigcap \mathcal{N}$, then for every $B_x \in \mathcal{N}$, $z \in B_x$ and by what we have just shown, $B_z \subseteq B_x$. This implies that $B_z \subseteq \bigcap \mathcal{N}$, which proves (SC3).

The last assertion of the lemma is clear. □

6. Ultrametric attractor theorems

In this section, we present a generalization of the attractor theorem of [4] to ultrametric spaces with partially ordered value sets, and show how to derive it from Theorem 7.

Take ultrametric spaces (X,d) and (X',d') and a function $\varphi : X \to X'$. An element $z' \in X'$ is called **attractor for** φ if for every $x \in X$ such that $z' \neq \varphi x$, there is an element $y \in X$ which satisfies:

(UAT1) $d'(\varphi y, z') < d'(\varphi x, z')$,
(UAT2) $\varphi(B(x,y)) \subseteq B(\varphi x, z')$,
(UAT3) if $t \in X$ such that $d'(\varphi x, z') < d'(\varphi t, z')$ and $\varphi(B(t,x)) \subseteq B(\varphi t, z')$, then $d(t,x)$ and $d(x,y)$ are comparable.

Condition (UAT1) says that the approximation φx of z' from within the image of φ can be improved, and condition (UAT2) says that this can be done in a somewhat continuous way. Condition (UAT3) is always satisfied when the value set of (X,d) is totally ordered, which implies that any two balls with nonempty intersection are comparable by inclusion. For this reason, it does not appear as a condition in the attractor theorem of [4]. But if the value set of (X,d) is not totally ordered, then it can happen that several "parallel universes" exist around a point; (UAT3) then guarantees that we can keep our approximations to remain in the same universe.

Theorem 15. *Assume that $z' \in X'$ is an attractor for $\varphi : X \to X'$ and that (X,d) is spherically complete. Then $z' \in \varphi(X)$.*

Proof: For $x \in X$, we define $B'_x := B'(\varphi x, z')$. Then we define a function $f : X \to X$ as follows. If $\varphi x = z'$, we set $fx = x$.

Now assume that $\varphi x \neq z'$. We choose some $y \in X$ that satisfies (UAT1), (UAT2) and (UAT3); then we set $fx := y$ and $B_x := B(x, fx)$. We have that $z' \in B'_x$ by definition, and $\varphi(B_x) = \varphi(B(x,fx)) \subseteq B(\varphi x, z') = B'_x$ holds by (UAT2); hence (AT1) is satisfied.

By (UAT1), we have that $d'(\varphi fx, z') < d'(\varphi x, z')$, which by (2) implies that $B'_{fx} = B(\varphi fx, z') \subsetneqq B(\varphi x, z') = B'_x$. Thus, (AT2) holds. We also see that $\varphi x \notin B(\varphi fx, z')$.

We run the above construction again with fx in place of x to obtain f^2x. By what we have already shown, $\varphi(B_{fx}) \subseteq B'_{fx}$, so $x \notin B_{fx}$. From (UAT3), where we replace t, x, y by x, fx, f^2x, we infer that $d(x, fx)$ and $d(fx, f^2x)$ are comparable. Therefore, $B_x \subseteq B_{fx}$ or $B_{fx} \subseteq B_x$. Since $x \notin B_{fx}$, it follows that $B_{fx} \subsetneqq B_x$, which proves that f is strictly contracting on orbits.

We have shown that z' is a weak f-attractor for φ. Our theorem will thus follow from Theorem 7 once we have proved that f also satisfies (SC3). Take an f-nest $\mathcal{N}$ and some $z \in \bigcap \mathcal{N}$. We have to show that $B_z \subseteq \bigcap \mathcal{N}$, that is, $B_z \subseteq B_x$ for all $B_x \in \mathcal{N}$. Since $z \in \bigcap \mathcal{N} \subseteq B_{fx}$, we have that

$$\varphi z \in \varphi(B_{fx}) \subseteq B'_{fx} = B(\varphi fx, z') .$$

It follows that

$$\varphi(B_z) \subseteq B'_z = B(\varphi z, z') \subseteq B(\varphi fx, z');.$$

But we have already shown that $\varphi x \notin B(\varphi f x, z')$, hence $x \notin B_z$. We have that

$$d'(\varphi z, z') \;\le\; d'(\varphi f x, z') \;<\; d'(\varphi x, z') \tag{6}$$

and

$$\varphi(B(x,z)) \;\subseteq\; \varphi(B_x) \;\subseteq\; B'_x \;=\; B(\varphi x, z')\;. \tag{7}$$

From (UAT3), where we replace t, x, y by x, z, fz, we infer that $d(x,z)$ and $d(z, fz)$ are comparable. We find that $B(x,z) \subseteq B(z, fz)$ or $B(z, fz) \subseteq B(x,z)$. Since $x \notin B_z$, we obtain that $B_z \subseteq B(x,z) \subseteq B_x$. □

Note that our condition (UAT3) is somewhat stronger than condition (8) of [PR] because we do not start with a given function f (which is called g in [PR]), but construct it in our proof. Rewritten in our present notation, condition (8) of [PR] states:

(UAT3′) if $d'(\varphi z, z') < d'(\varphi x, z')$ and $B(z, fz) \cap B(x, fx) \neq \emptyset$, then $d(z, fz) < d(x, fx)$.

We show how (UAT3′) implies (SC3). For $z \in \bigcap \mathcal{N}$, we have to show that $B_z \subseteq B_x$ for all $B_x \in \mathcal{N}$. As in the above proof, one shows that (6) holds. Since $z \in \bigcap \mathcal{N} \subseteq B_x = B(x, fx)$, we have that $B(z, fz) \cap B(x, fx) \neq \emptyset$. Hence by (UAT3′), $d(z, fz) < d(x, fx)$, which by (1) implies that $B_z \subseteq B_x$. Observe that the condition "$d(z, fz) < d(x, fx)$" in (UAT3′) can be replaced by the weaker condition that $d(z, fz)$ and $d(x, fx)$ are comparable.

7. Completeness by stages

In this section, we consider valued fields (K, v). In order to be compatible with the way we have presented ultrametrics in the previous sections, we write the valuation v multiplicatively, that is, the value group is a multiplicatively written ordered abelian group with neutral element $1 = v(1)$, and we add a smallest element $0 = v(0)$. The axioms for a valuation in this notation are:

(VF1) $v(x) = 0 \Leftrightarrow x = 0$,
(VF2) $v(xy) = v(x)v(y)$,
(VF3) $v(x + y) \le \max\{v(x), v(y)\}$.

The underlying ultrametric is obtained by setting $d(x, y) := v(x - y)$.

We will work in the valuation ideal $\mathcal{M} = \{x \in K \mid v(x) < 1\}$ of (K, v) since this facilitates the notation, and the typical applications of the fixed point theorem we are going to prove can be made to deal with functions $f : \mathcal{M} \to \mathcal{M}$.

We assume the reader to be familiar with the theory of pseudo Cauchy sequences (see for instance [3]). P. Ribenboim introduced in [9] the notion of **distinguished pseudo Cauchy sequence**. For a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in $\mathcal{M}$, indexed by a limit ordinal λ, the original definition of "distinguished" is equivalent to the following: for every $\mu < \lambda$ there is $\nu < \lambda$ such that

$$v(a_{\nu+1} - a_\nu) \leq v(a_{\mu+1} - a_\mu)^2 \ .$$

The valued field (K, v) is called **complete by stages** if every distinguished pseudo Cauchy sequence in $\mathcal{M}$ has a pseudo limit in K. Ribenboim proves in [9] that every such field is henselian. For this proof, one can use a theorem like the following:

Theorem 16. *Take a valued field (K, v) that is complete by stages and a contracting function $f : \mathcal{M} \to \mathcal{M}$. If for every $x \in K$ there is $j \in \mathbb{N}$ such that*

$$v(f^j x - f^{j+1} x) \ \leq \ v(x - fx)^2 \ ,$$

then f has a fixed point in K.

Note that $v(x - fx) < 1$ since $x, fx \in \mathcal{M}$. In the theorem, $\mathcal{M}$ can be replaced by any ultrametric ball B in K as long as $v(x - fx) < 1$ for all $x \in B$.

In order to deduce this theorem from Theorem 6, we need to show the connection between completeness by stages and nests of balls with certain properties. We will call a nest $\mathcal{N}$ of ultrametric balls in $\mathcal{M}$ **distinguished** if for all $x, y \in \mathcal{M}$ with $B(x, y) \in \mathcal{N}$ there are $x', y' \in \mathcal{M}$ such that $v(x' - y') \leq v(x - y)^2$ and $B(x', y') \in \mathcal{N}$. Then the following holds:

Lemma 17. *A valued field (K, v) with valuation ideal $\mathcal{M}$ is complete by stages if and only if each distinguished nest of ultrametric balls in $\mathcal{M}$ has a nonempty intersection.*

In this lemma, $\mathcal{M}$ can be replaced by the valuation ring $\mathcal{O}$, and also by the ultrametric balls $a + \mathcal{M}$ and $a + \mathcal{O}$ for all $a \in K$. The proof of the lemma is similar to the proof of the fact that an ultrametric space is spherically complete if and only if every pseudo Cauchy sequence in this space has a pseudo limit. It is based on the following easy observation:

Lemma 18. *Every nest of balls without a smallest ball admits, in the ordering given by reverse inclusion, a cofinal well ordered subnest.*

Now the method of proof is to associate such a coinitial well ordered subnest with a pseudo Cauchy sequence such that each pseudo limit will lie in the intersection of the nest — and vice versa. Since every subnest of a distinguished nest is again distinguished, this will give rise to a distinguished pseudo Cauchy sequence. Conversely, if $(a_\nu)_{\nu<\lambda}$ is a distinguished pseudo Cauchy sequence in $\mathcal{M}$, then $\{B(a_\nu, a_{\nu+1}) \mid \nu < \lambda\}$ is a distinguished nest of balls in $\mathcal{M}$.

Proof of Theorem 16: As before, we set $B_x := B(x, fx)$ and note that $x \in B_x$. Since f is contracting, we have that $v(fx - f^2 x) \leq v(x - fx)$, which by (1) implies that $B_{fx} \subseteq B_x$. If $fx \neq x$, then by assumption, there is $i \in \mathbb{N}$ such that $v(f^i x - f^{i+1}x) \leq v(x-fx)^2 < v(x-fx)$; this yields that $x \notin B_{f^i x}$ and $B_{f^i x} \subsetneq B_x$. We have now proved that f is strongly contracting on orbits. Since f is contracting, this implies by way of Lemma 14 that f also satisfies (SC3).

Take an f-nest $\mathcal{N}$. Every ball in $\mathcal{N}$ is of the form $B(x,fx)$ and there is some $i \in \mathbb{N}$ such that $v(f^i x - f^{i+1}x) \leq v(x-fx)^2$. By the definition of an f-nest, $B_{f^i x} \in \mathcal{N}$. Thus, $\mathcal{N}$ is distinguished. Since (K,v) is assumed to be complete by stages, Lemma 17 shows that $\mathcal{N}$ has a nonempty intersection. Since f also satisfies (SC3) and is strongly contracting on orbits, this implies that the conditions of Theorem 4 are satisfied, and we obtain the assertion of Theorem 16.

8. Banach's Fixed Point Theorem

Banach's Fixed Point Theorem states that every strictly contracting function on a complete metric space (X,d) has a unique fixed point. A function $f: X \to X$ is called **strictly contracting** if there is a positive real number $C < 1$ such that $d(fx,fy) \leq Cd(x,y)$ for all $x,y \in X$. We will show now how Banach's Fixed Point Theorem fits into the setting of Theorems 4 and 6. We work in the ball space $(X,\mathcal{B})$ where $\mathcal{B}$ consists of all balls $\{y \in X \mid d(x,y) \leq r\}$ for $x \in X$ and $r \in \mathbb{R}^{\geq 0}$. This ball space is spherically complete since (X,d) is complete.

We will prove the existence of fixed points under the slightly more general assumption that f is

1) **contracting**, that is, $d(fx,fy) \leq d(x,y)$ for all $x,y \in X$, and
2) **strictly contracting on orbits**, that is, there is a positive real number $C < 1$ such that $d(fx,f^2x) \leq Cd(x,fx)$ for all $x \in X$.

Take any $x \in X$. Then

$$\begin{aligned} d(x,f^i x) &\leq d(x,fx) + d(fx,f^2x) + \ldots + d(f^{i-1}x,f^i x) \\ &\leq d(x,fx)(1 + C + C^2 + \ldots + C^{i-1}) \\ &\leq d(x,fx)\sum_{i=0}^{\infty} C^i \;=\; \frac{d(x,fx)}{1-C}\,. \end{aligned}$$

Hence if we set

$$B_x \;:=\; \left\{ y \in X \mid d(x,y) \;\leq\; \frac{d(x,fx)}{1-C} \right\}\,,$$

then $f^i x \in B_x$ for $i \geq 0$. In particular, $x \in B_x$, hence (SC1) holds.

We wish to show that $B_{fx} \subseteq B_x$. Take any $y \in B_{fx}$. Then

$$\begin{aligned} d(x,y) &\leq d(x,fx) \,+\, d(fx,y) \;\leq\; d(x,fx) \,+\, \frac{d(fx,f^2x)}{1-C} \\ &\leq d(x,fx) \,+\, \frac{C}{1-C}\, d(x,fx) \;=\; \frac{d(x,fx)}{1-C}\,. \end{aligned}$$

Thus, $y \in B_x$, which proves our assertion.

Since $C < 1$, there is some $i \geq 1$ such that

$$\frac{C^i}{1-C} < \frac{1}{2} .$$

Then

$$\frac{d(f^i x, f^{i+1}x)}{1-C} \leq \frac{C^i}{1-C}\, d(x,fx) < \frac{1}{2}\, d(x,fx) ,$$

which implies that x and fx cannot both lie in $B_{f^i x}$. Therefore, $B_{f^i x} \subsetneq B_x$ and we have now proved that (SC2) holds.

Next, we show that also (SC3) holds. Take an f-nest $\mathcal{N}$ and assume that $z \in \bigcap \mathcal{N}$. Pick any $B_x \in \mathcal{N}$ and $i > 0$. Since f is contracting, $d(f^i x, fz) \leq d(f^{i-1}x, z) = d(z, f^{i-1}x)$. Using that $z \in \mathcal{N} \subseteq B_{f^i x}$ for all i, we compute:

$$\begin{aligned} d(z,fz) &\leq d(z,f^i x) + d(f^i x, fz) \leq d(z,f^i x) + d(z, f^{i-1}x) \\ &\leq \frac{d(f^i x, f^{i+1}x)}{1-C} + \frac{d(f^{i-1}x, f^i x)}{1-C} \\ &\leq \frac{C^i}{1-C}\, d(x,fx) + \frac{C^{i-1}}{1-C}\, d(x,fx) = C^{i-1}\frac{C+1}{1-C}\, d(x,fx) . \end{aligned}$$

Since $\lim_{i\to\infty} C^i = 0$, we obtain that $fz = z$, so we have found a fixed point. It follows that $B_z \subseteq \bigcap \mathcal{N}$ (in fact, $\bigcap \mathcal{N} = \{z\} = B_z$). This shows that f is self-contractive.

Note that if f is strictly contracting, then the fixed point is unique. Indeed, if there were distinct fixed points x, y, then $d(x,y) = d(fx,fy) < d(x,y)$, a contradiction.

9. The case of ordered abelian groups and fields

In this section we will discuss various forms of fixed point theorems in the case of ordered abelian groups and fields. Here, we always mean that the ordering is total. The ordering induces a natural valuation; we will recall its definition in Section 9.1. This valuation is nontrivial if and only if the ordering is nonarchimedean. Since the valuation induces an ultrametric, our ultrametric fixed point theorems can be translated to the present case. We will do this in Section 9.2.

However, the most natural idea to derive a ball space from the ordering of an ordered abelian group $(G,<)$ is to define the **order balls** in G to be the sets of the form

$$B_o(g;r) := \{z \in G \mid |g-z| \leq r\}$$

for arbitrary $g \in G$ and nonnegative $r \in G$. We set

$$\mathcal{B}_o := \{B_o(g;r) \mid g \in G,\, 0 \leq r \in G\} .$$

But it turns out that working with these balls leads to complications. We will discuss them in Section 9.3 and discuss some ways to avoid them in Sections 9.4 and 9.5.

Before we continue, we need some preliminaries and general background.

9.1. **Preliminaries on non-archimedean ordered abelian groups and fields.** Take an ordered abelian group $(G,<)$. Two elements $a,b \in G$ are called **archimedean equivalent** if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. The ordered group $(G,<)$ is archimedean ordered if all nonzero elements are archimedean equivalent. If $0 \leq a < b$ and $na < b$ for all $n \in \mathbb{N}$, then "a is infinitesimally smaller than b" and we will write $a \ll b$.

We define the **natural valuation** of $(G,<)$ as follows. We denote by va the archimedean equivalence class of a. The set of archimedean equivalence classes is ordered as follows: $va < vb$ if and only if $|a| < |b|$ and a and b are not archimedean equivalent, that is, if $n|a| < |b|$ for all $n \in \mathbb{N}$. We write $0 := v0$; this is the minimal element in the totally ordered set of equivalence classes. The function $a \mapsto va$ is a group valuation on G, i.e., it satisfies $vx = 0 \Leftrightarrow x = 0$ and the ultrametric triangle law

(UT) $v(x-y) \leq \max\{vx, vy\}$.

The natural valuation induces an ultrametric defined by

$$d(x,y) := v(x-y)$$

and hence an ultrametric ball space, with the balls $B_u(x,y)$ defined as in Section 5. We will call this ball space the **natural ultrametric ball space** and denote it by $\mathcal{B}_u(G,<)$. Note that all ultrametric balls are cosets of convex subgroups in G.

Take any element $a \in G$, $a \neq 0$. Then $\mathcal{O}_a := \{g \in G \mid vg \leq va\}$ and $\mathcal{M}_a := \{g \in G \mid vg < va\}$ are convex subgroups of G. It follows that $C_a := \mathcal{O}_a/\mathcal{M}_a$ is an archimedean ordered group, called the **archimedean component of** G **at** a. By the Theorem of Hölder, we may therefore consider it as a subgroup of $\mathbb{R}$.

If $(K,<)$ is an ordered field, then we consider the natural valuation on its ordered additive group and define $va \cdot vb := v(ab)$. This turns the set of archimedean classes into a multiplicatively written ordered abelian group, with neutral element $1 := v1$ and inverses $(va)^{-1} = v(a^{-1})$. In this way, v becomes a field valuation (with multiplicatively written value group). It is the finest valuation on K which is compatible with the ordering.

Given a totally ordered index set Γ and for every $\gamma \in \Gamma$ an arbitrary abelian group C_γ, we define a group called the **Hahn product**, denoted by $\mathbf{H}_{\gamma\in\Gamma}\, C_\gamma$. Consider the product $\prod_{\gamma\in\Gamma} C_\gamma$ and an element $c = (c_\gamma)_{\gamma\in\Gamma}$ of this group. Then the **support of** c is the set of all $\gamma \in \Gamma$ for which $c_\gamma \neq 0$. As a set, the Hahn product is the subset of $\prod_{\gamma\in\Gamma} C_\gamma$ containing all elements whose support is an anti-wellordered subset of Γ, that is, every nonempty subset of the support has a maximal element. In particular, the support of every element c in the Hahn product has a maximal element γ_0, which enables us to define a group valuation by setting $vc = \gamma_0$. The Hahn product is a subgroup of the product group. Indeed, the support of the sum of two elements is contained in the union of their supports, and the union of two anti-wellordered sets is again anti-wellordered.

If the components C_γ are (not necessarily archimedean) ordered abelian groups, we obtain the **ordered Hahn product**, also called **lexicographic product**, where the ordering is defined as follows. Given a nonzero element $c = (c_\gamma)_{\gamma\in\Gamma}$, let γ_0 be the maximal element of its support. Then we take $c > 0$ if and only if $c_{\gamma_0} > 0$. If all C_γ are archimedean ordered, then the valuation v of the Hahn product coincides with the natural valuation of the ordered Hahn product. Every ordered abelian group G can be embedded in the Hahn product with its set of archimedean classes of nonzero elements as index set and its archimedean components as components. Then G is spherically complete w.r.t. the ultrametric balls if and only if the embedding is onto.

In the case of an ordered field K, we know from [3, Theorem 6] that K can be embedded in the power series field with exponents in the value group and coefficients in the residue field of its natural valuation. (A nontrivial factor set may be needed, unless the positive part of the residue field is closed under radicals, which for instance is the case if K is real closed.) Again, K is spherically complete w.r.t. the ultrametric balls if and only if the embedding is onto. If we forget about the multiplication on the power series field, we obtain the underlying Hahn product that corresponds to the ordered additive group of K. The multiplication of K induces an isomorphism of all its components to the additive ordered group of the residue field of K.

For more information on the notions we will work with, we refer the reader to chapter 2 of [5]. But note that there, valuations are written in the Krull style, which inverts the ordering on the (additively written) value groups, and this also affects the definition of the Hahn product.

9.2. **Ultrametric balls.** Via the natural valuation, the ultrametric fixed point theorems provide fixed point theorems for ordered abelian groups and fields. A valuation is called **spherically complete** if its associated ultrametric ball space is spherically complete, and it is called **complete by stages** if its associated ultrametric ball space is complete by stages.

Take an ordered abelian group G and a function $f : G \to G$. It will be called **o-contracting** if

$$|fx - f^2x| \leq |x - fx|$$

for all $x \in G$; note that an o-contracting function is also contracting in the ultrametric sense. The property

$$x \neq fx \implies \exists i \geq 1 : \; |f^ix - f^{i+1}x| \; \ll \; |x - fx| \tag{8}$$

implies the property (3). Hence, the following theorem is an immediate consequence of Theorem 13. In order to obtain it for an ordered field K, take G to be the additive group of K.

Theorem 19. *Take an ordered abelian group G whose natural valuation is spherically complete. If $f : G \to G$ is an o-contracting function that satisfies (8), then it has a fixed point.*

Take an ordered field K and let v denote its natural valuation. Then the valuation ideal $\mathcal{M}$ of v is the set of all infinitesimals of K, that is, the elements $x \in K$ such that $|x| \ll 1$. The next theorem follows directly from Theorem 16. We suspect that the theorem becomes false if $\mathcal{M}$ is replaced by the valuation ring $\mathcal{O}$.

Theorem 20. *Take an ordered field K with $\mathcal{M}$ the set of its infinitesimals, whose natural valuation is complete by stages. If $f : \mathcal{M} \to \mathcal{M}$ is an o-contracting function and for every $x \in \mathcal{M}$ there is $j \in \mathbb{N}$ such that*

$$|f^j x - f^{j+1} x| \;\leq\; |x - fx|^2 \,,$$

then f has a fixed point in $\mathcal{M}$.

9.3. **Order balls.** Working with groups or fields that have spherically complete underlying order ball spaces is of little use as they are very close, if not equal, to being cut complete. But all cut complete ordered abelian groups are isomorphic to $\mathbb{Z}$ or $\mathbb{R}$, and all cut complete ordered fields are isomorphic to $\mathbb{R}$. We wish to give the reader an idea of the basic arguments involved.

If we want to use the spherical completeness of the underlying order ball space for showing cut completeness, we have to be able to “zoom in” on a cut by a nest of order balls. For simplicity, let us assume that G is 2-divisible. Then every closed interval $[a, b]$ is an order ball:

$$[a, b] \;=\; B_o\left(\frac{a+b}{2}; \frac{|b-a|}{2}\right) \,.$$

We consider Dedekind cuts (Λ^L, Λ^R) in G and call them **cardinality-symmetric** if the cofinality of the left cut set Λ^L equals the coinitiality of the right cut set Λ^R.

Lemma 21. *Take a cut (Λ^L, Λ^R) in the 2-divisible ordered abelian group $(G, <)$. Then the cut is cardinality-symmetric if and only if there is a nest of order balls with empty intersection where all balls in the nest have nonempty intersection with both cut sets.*

Proof: Denote by κ_L the cofinality of Λ^L and by κ_R the coinitiality of Λ^R. Choose a cofinal sequence $(a_\nu)_{\nu<\kappa_L}$ in Λ^L and a coinitial sequence $(b_\nu)_{\nu<\kappa_R}$ in Λ^R. If $\kappa_L = \kappa_R$, then $\{[a_\nu, b_\nu] \mid \nu < \kappa_L\}$ is a nest of order balls with empty intersection.

Now assume that $\kappa_L < \kappa_R$ and that $(B_\mu)_{\mu<\kappa}$ is a descending chain of order balls which have nonempty intersection with both Λ^L and Λ^R.

If $\kappa < \kappa_R$, then for large enough $\nu < \kappa_R$, the element b_ν is contained in all B_μ and the intersection over the chain is nonempty. If $\kappa \geq \kappa_R > \kappa_L$, then for some $\nu < \kappa_L$, a_ν is contained in all B_μ and again the intersection over the chain is nonempty.

If $\kappa_L > \kappa_R$, then a symmetrical argument shows that the intersection over the chain is nonempty. In view of Lemma 18, we have now proved that if $\kappa_L \neq \kappa_R$, then every nest of order balls all of which have nonempty intersection with both cut sets, has a nonempty intersection. □

Consequently, a 2-divisible ordered abelian group with spherically complete underlying order ball space cannot admit a cardinality-symmetric cut. The same holds for $\mathbb{R}$, as for every Dedekind cut either the lower cut set has a maximal element and thus cofinality 1, or the upper cut set has a smallest element and thus coinitiality 1. Every proper subgroup of $\mathbb{R}$ has symmetric cuts, with cofinality and coinitiality $\aleph_0$ if it is dense, and with cofinality and coinitiality 1 if it is discrete. So it remains to consider non-archimedean ordered groups. In this case, our problem is related, via their natural valuations, to the following

Open Problem: Does every (totally) ordered set admit a cardinality-symmetric cut, or a cut where one of the cardinalities is countable and the other is 1?

If this question has an affirmative answer, then it can be shown that every 2-divisible ordered abelian group and hence also every ordered field with spherically complete underlying order ball space is isomorphic to $\mathbb{R}$. As a consequence, the result can be proven when the natural valuation of the group or the field takes at most countably many values.

We see that for a fixed point theorem working with order balls to be interesting, we should restrict it to ball spaces that are significantly smaller than the order ball space. We will discuss possible approaches in the next sections.

9.4. **Hybrid ball spaces.** One idea for involving order balls in a fixed point theorem is to enlarge the underlying ultrametric ball space of an ordered abelian group by an "admissible" set of order balls. We will make use of two very easy principles:

Lemma 22. *1) If the ball space $(X,\mathcal{B})$ is spherically complete and $\mathcal{B}' \subseteq \mathcal{B}$, then also $(X,\mathcal{B}')$ is spherically complete.*

2) If $\mathcal{B}_i$, $1 \leq i \leq n$, are collections of subsets of X, then the ball space $(X, \bigcup_{i=1}^n \mathcal{B}_i)$ is spherically complete if and only if all of the ball spaces $(X,\mathcal{B}_i)$, $1 \leq i \leq n$, are spherically complete.

We leave the easy proofs to the reader. Just note that for every nest of balls in $(X, \bigcup_{i=1}^n \mathcal{B}_i)$ there must be an i and a cofinal (under reverse inclusion) subnest of balls that all lie in $\mathcal{B}_i$.

We set

$$\mathcal{B}_h(G,<) \;:=\; \mathcal{B}_u(G,<) \,\cup\, \{B_o(g;q|g|) \mid g\in G,\, 0<q\in\mathbb{Q}\}$$

and call it the **hybrid ball space of** $(G,<)$. In order to compute with the product $q|g|$ we pass to the divisible hull of G with its unique extension of the ordering. The balls $B_o(g;q|g|)$ in G are then the restrictions to G of the corresponding balls in the divisible hull.

We wish to prove a characterization of those ordered abelian groups and ordered fields which have a spherically complete hybrid ball space. We need a few preparations.

Lemma 23. *The integers and the reals with the canonical ordering are spherically complete w.r.t. the order balls.*

Proof: First, observe that every nest of order balls in the integers contains a smallest ball, so the integers are spherically complete w.r.t. the order balls.

Next, take a nest $\mathcal{N}$ of order balls in the reals. Pick a ball B in this nest and consider the nest $\mathcal{N}_0$ of all balls in $\mathcal{N}$ that are contained in B; this nest has the same intersection as $\mathcal{N}$. The order ball B is compact in the order topology of $\mathbb{R}$. Hence by Lemma 9, the intersection of $\mathcal{N}_0$ is nonempty. This proves that the reals are spherically complete w.r.t. the order balls. □

Lemma 24. *An ordered abelian group G is spherically complete w.r.t.*

$$\mathcal{B}_a \;:=\; \{B_o(g;q|g|) \mid g\in G,\, 0<q\in\mathbb{Q}\}$$

if and only if all of its archimedean components are isomorphic to $\mathbb{Z}$ or $\mathbb{R}$.

Proof: Assume that the ordered abelian group G is spherically complete w.r.t. $\mathcal{B}_a$. Take any nonzero $a\in G$ and consider the corresponding archimedean component $C_a=\mathcal{O}_a/\mathcal{M}_a\subseteq\mathbb{R}$. Denote by $\mathcal{O}_a\ni g\mapsto\overline{g}$ the canonical epimorphism from $\mathcal{O}_a$ to C_a, and assume it extended to the divisible hull of G.

Suppose that C_a is not isomorphic to $\mathbb{Z}$ or $\mathbb{R}$. Then there is a cut (Λ^L,Λ^R) such that Λ^L does not have a maximal element and Λ^R does not have a minimal element. Then we can choose an increasing sequence of elements c_i, $i\in\mathbb{N}$, cofinal in Λ^L, and a sequence of positive rational numbers q_i such that $c_i+q_i|c_i|$, $i\in\mathbb{N}$, forms a coinitial decreasing sequence in Λ^R. We choose elements $b_i\in\mathcal{O}_a$ with $c_i=\overline{b_i}$. Then $\overline{b_i+q_i|b_i|}=c_i+q_i|c_i|$. It follows that the balls $B_o(b_i;q_i|b_i|)$ form a nest in G. If its intersection would contain an element b, then $c=\overline{b}$ would satisfy $c_i\le c\le c_i+q_i|c_i|$ for all $i\in\mathbb{N}$, which is impossible. This contradiction proves that if G is spherically complete w.r.t. $\mathcal{B}_a$, then all of its archimedean components are isomorphic to $\mathbb{Z}$ or $\mathbb{R}$.

For the converse, take a nest $\{B_o(g_i;q_i|g_i|)\mid i\in I\}$ of balls in $\mathcal{B}_a$. We distinguish two cases.

First, assume that the set vg_i, $i \in I$, has no smallest element. Then for every $i \in I$ there is $j \in I$ such that $B_o(g_j; q_j|g_j|) \subseteq B_o(g_i; q_i|g_i|)$ and $vg_j < vg_i$. But if $q_i < 1$ then there is no element in $B_o(g_i; q_i|g_i|)$ of value $< vg_i$. So we find that $q_i \geq 1$, which implies that $0 \in B_o(g_i; q_i|g_i|)$. It follows that 0 lies in all balls of the nest and hence also in its intersection.

Next, assume that the set $\{vg_i \mid i \in I\}$ has a smallest element vg_{i_0}. Set $g = g_{i_0}$ and $I_0 := \{i \in I \mid vg_i = vg\}$. Note that $B_o(g'; q'|g'|) \subseteq B_o(g''; q''|g''|)$ implies that $q'|g'| \leq q''|g''|$, which in turn implies that $vg' \leq vg''$. Hence if $i \in I_0$ and $j \in I \setminus I_0$, then $B_o(g_i; q_i|g_i|) \subseteq B_o(g_j; q_j|g_j|)$. Therefore, the intersection of the nest $\{B_o(g_i; q_i|g_i|) \mid i \in I\}$ is equal to that of the nest $\{B_o(g_i; q_i|g_i|) \mid i \in I_0\}$. All elements in these balls have v-value $\leq vg$, so we can project the balls into the archimedean component C_g, where the images of the balls will form a nest $\{B_o(\overline{g_i}; \overline{q_i|g_i|}) \mid i \in I_0\}$ of order balls. By assumption, C_g is isomorphic to the ordered groups $\mathbb{Z}$ or $\mathbb{R}$. By Lemma 23, the induced nest has a nonempty intersection. Hence the same holds for the nest $\{B_o(g_i; q_i|g_i|) \mid i \in I\}$. □

From Lemma 24 and part 2) of Lemma 22, we now obtain:

Proposition 25. *An ordered abelian group is spherically complete w.r.t. the hybrid balls if and only if it is spherically complete w.r.t. the ultrametric balls and all of its archimedean components are isomorphic to the ordered additive groups of* $\mathbb{Z}$ *or* $\mathbb{R}$.

The archimedean components of the ordered additive group of an ordered field are all isomorphic to its residue field under the natural valuation, so they cannot be isomorphic to $\mathbb{Z}$. From Proposition 25 together with the facts on Hahn products and power series fields that we have outlined in Sextion 9.1, we obtain the following theorem:

Theorem 26. *An ordered abelian group has a spherically complete hybrid ball space if and only if it is isomorphic to a Hahn product with all its components being* $\mathbb{Z}$ *or* $\mathbb{R}$.

An ordered field has a spherically complete hybrid ball space if and only if it is isomorphic to a power series field with residue field $\mathbb{R}$.

9.5. **Restricted order ball spaces.** In this section we show how to obtain restricted order ball spaces on ordered Hahn products $G = \mathbf{H}_{\gamma\in\Gamma} C_\gamma$, where all components C_γ are archimedean ordered abelian groups. We take $\mathcal{B}_r(G, <)$ to be the collection of all order balls $B_o(g; r)$ where the support of r is a singleton $\{\gamma\}$ and γ is the minimum of the support of g (not the maximum!). We leave it to the reader to prove, along the lines of the previous section, the following result:

Proposition 27. *The restricted order ball space* $\mathcal{B}_r(G, <)$ *is spherically complete if and only if all components* C_γ *are isomorphic to* $\mathbb{Z}$ *or* $\mathbb{Q}$.

In order to understand the above restriction we present an example that was suggested to us by our referee. Consider the lexicographic product $\mathbb{R}\times\mathbb{R}$, that is, $(0,1)\ll(1,0)$. The nest of order balls

$$\{B_o((s,0);(s,-1/s)\mid 0<s\in\mathbb{R}\}$$

has empty intersection. In fact, it zooms in on the cut whose lower cut set is $\{(s,t)\mid 0\geq s\in\mathbb{R},\, t\in\mathbb{R}\}$, which means that the cut is the upper edge of the convex subgroup $\{0\}\times\mathbb{R}$. Note that the same holds for the nests $\{B_o((s,1/s);(s,0)\mid 0<s\in\mathbb{R}\}$ and $\{B_o((s,1/s);(s,-1/s)\mid 0<s\in\mathbb{R}\}$. All of these nests are built from balls for which the above conditions on the supports are not met, which is crucial for the construction.

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